Clique-width and well-quasi-order
Case for Support

Previous Track Record

Robert Brignall (PI) has been a Lecturer in Combinatorics at The Open University since 2010. He received his PhD in 2007 from the University of St Andrews, and from 2007–2010 he was a Heilbronn Research Fellow at The University of Bristol. In Bristol, he spent 50% of his time on classified research directed by the Heilbronn Institute, and 50% on his own research agenda. He supervises one PhD student who is on track to finish in the first half of 2015 (within three years of the start of his PhD), and he supervised a Research Assistant from 2012–14, part-funded by EPSRC grant EP/J006130/1, who now holds a permanent academic position in the UK. He has written 17 peer-reviewed papers, with a further 2 currently under review.

His career began in the structural study of permutation classes, and he continues to make lasting contributions to this area [10, 12, 13, 16, 18], and its consequences for the enumeration of permutation classes [1–4].

Following his PhD, his work expanded in two directions: first, he looked at the question of well-quasi-ordering for permutations, and this resulted in a single-author paper which represents the state-of-the-art in infinite antichain construction [11]. Second, he applied his structural expertise to the wider study of combinatorial structures [17], with a particular emphasis on the cross-fertilisation of results between permutations and graphs.

His research in these two directions were combined when in 2012 he became PI on grant EP/J006130/1. Through new collaborations with researchers in Warwick (including the named RA on this proposal), this grant catalysed the process of translating structural results from permutations to graphs [15], most especially with regards to well-quasi-ordering and infinite antichains [7].

The importance to the study of structure and well-quasi-ordering in graphs of the second subject of this proposal, clique-width, was brought to the attention of the PI during the course of this grant. Consequently, the PI undertook to enhance his intuition of this area, particularly in identifying the interface between graph classes where clique-width is bounded, and graph classes where clique-width is unbounded [6]. He also considered similar questions for a restricted version of this parameter called linear clique-width [14], which is important both to extend our understanding of the more general problem, and because it has direct relevance to the construction of infinite antichains.

Aistis Atminas (proposed RA) completed his PhD thesis [5], entitled 'Well-quasi-ordering of Combinatorial Structures', at the University of Warwick in November 2014. He already has 9 papers published or submitted for publication, and these include some of the results that underpin the methodology of this proposal, most notably [8] and [9]. He has an existing and effective working relationship with PI, through collaboration on two projects [6, 7].

Collaborators

Vadim Lozin is an Associate Professor (Reader) at DIMAP, University of Warwick. With well over 100 publications, he has direct expertise in both well-quasi-ordering and clique-width for graphs, which complements the PI's own knowledge well. He has been PI on two EPSRC grants since 2011 totalling over £500k. One of these grants concerned clique-width of graph classes, and the other, which is ongoing until 2017, seeks to lay theoretical foundations for a number of graph algorithms, including the 'maximum independent set' problem. In the past 2 years, the PI and Lozin have already collaborated successfully on three projects [6, 7, 15]

Vincent Vatter is an Associate Professor at the University of Florida, USA. His research interests relate very strongly to the PI's – he has over 40 papers in combinatorics, and 8 joint works with the PI [4, 7, 12–14, 16–18]. This ongoing collaboration, which has touched on many of the topics of this proposal, makes him a natural addition to this project.

Host institution

The Open University has a long-standing history in combinatorial research, made highly visible through the Winter Combinatorics Meeting which has run at the university for the past 15 years. Since 2011, the PI has been involved in the organisation of this ongoing meeting, and for the duration of this proposal he intends to use this meeting to invite key academics from overseas to speak, and to stay for a few days in order to have in-depth discussions.

The combinatorics research group, counting 11 members from postgraduates to emeritus professors, provides an ideal environment for this project. Individuals have a diverse range of interests, including (pertinent to this proposal) various aspects of graph theory, and the connections between model theory and infinite designs.

PI and RA References

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Description of Proposed Research

Background

Well-quasi-ordering and infinite antichains

Robertson and Seymour's celebrated Graph Minor Theorem [35] proves that in any infinite collection of graphs there exists one which is a minor of another. An *infinite antichain* in an ordered collection is an infinite set for which no pair of elements is comparable. Thus, the Graph Minor Theorem states that there are no infinite antichains of graphs when they are ordered by the minor relation. Consequently, graphs under the minor ordering are said to be *well-quasi-ordered*. The consequences of both the statement of the Graph Minor Theorem and the structure theory developed for its proof are extensive, particularly in the study of computational complexity and fixed parameter tractability (see, for example, [21, 27]). For example, one corollary is that the question "is a graph H a minor of G?" can be answered in polynomial time in $|G|$ [34].

Often, collections of graphs that satisfy specified properties form *downsets* within some ordering. That is, whenever a graph is in a downset, then so too are all the graphs that are 'less than' it in the ordering. A convenient way to describe such downsets is by the unique set of *minimal forbidden graphs*. For example, Wagner's Theorem tells us that in the minor ordering the planar graphs are a downset defined by forbidding two graphs: no planar graph contains either the complete graph K_5 or the complete bipartite graph $K_{3,3}$, and every graph which is not planar *must* contain at least one of these two graphs as a minor.

For orderings other than the graph minor ordering, infinite antichains do exist. Two notable examples are the subgraph and induced subgraph orderings – Figure 1 illustrates two sets of graphs which form infinite antichains in both of these orderings. However, this does not mean that every downset in these orderings must contain an infinite antichain. In fact, Ding [25] showed that if a downset in the subgraph order does not have infinite intersection with either of the two antichains illustrated in Figure 1, then that downset is well-quasi-ordered.

The situation for the induced subgraph ordering is much less straightforward, as there are a wide variety of antichains that may need to be considered. Nevertheless, gaining a better un-

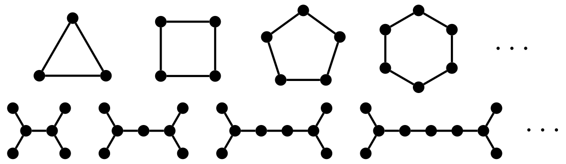

Figure 1: The first four graphs in two infinite antichains of graphs: in each case, no vertices or edges can be deleted from a larger graph to form a smaller one.

derstanding of this ordering is important: as well as being a natural one to consider mathematically, the induced subgraph ordering is the one best equipped to handle collections of graphs with a high edge density. Moreover, the 'clique-width' graph parameter respects the induced subgraph ordering, and on downsets where this parameter is bounded there are some far-reaching algorithmic implications (see later). A downset in this ordering is often called a *hereditary property*, though here, as elsewhere, we will use the term *(graph) class*.

As with other orderings, the set of minimal forbidden elements is a common way to define graph classes, particularly as input for algorithms. Whereas in the minor ordering such minimal forbidden sets must be finite (an important consequence of the Graph Minor Theorem), in the induced subgraph ordering this is not always the case. However, for the purpose of algorithmic development, we will where necessary restrict our attention to classes that are *finitely defined* – i.e. that have a finite set of minimal forbidden elements. On such classes, the following decision problem is of crucial importance.

Question 1. *Given a finitely defined graph class, is it well-quasi-ordered?*

Specific instances or approaches to answer this question have been considered by multiple authors over the years (see, e.g., [23, 24, 26, 30, 32]), but one of our two major objectives is to give a complete decision procedure to answer this question.

To prove a class is well-quasi-ordered, classical tools such as Higman's Theorem [28] and the 'minimal bad sequence' argument pioneered by Nash-Williams [31] can be applied, if a suitable structural description of the graphs in the class is known. This structural characterisation itself can enable the development of effective algorithms.

On the other hand, an increased understanding of the infinite antichains that cause a class to fail to be well-quasi-ordered can also indicate structurally how well-behaved the class is. For example, some classes contain a *canonical* antichain [26], whereby a subclass is well-quasi-ordered if and only if it has only finite intersection with that antichain.

The clique-width graph parameter

Formally, a *graph parameter* is any function that takes as input a graph, and outputs some number. The parameters of interest here are those that give some measurement of structure, such as 'tree-width' (made famous through its role in the proof of the Graph Minor Theorem). Specifically, Robertson and Seymour distinguish between the case where all the graphs in a downset in the minor ordering have tree-width bounded by some k, and the case where the tree-width is unbounded. Much of the whole proof of the Graph Minor Theorem is devoted to a structure theorem to handle the latter case.

A more recent parameter is *clique-width*, which measures the structural complexity of a graph when it is constructed by means of four elementary operations on the vertices. Clique-width is a generalisation of tree-width, in the sense that a graph with bounded tree-width also has bounded clique-width. In the study of algorithms, Courcelle, Makowsky and Rotics [22] showed that a large number of graph algorithms which are NP-hard in general can be solved in linear time for classes where all the graphs have clique-width at most some fixed k – this is an instance of *fixed parameter tractability*. For a survey of clique-width, see [29].

The connection between clique-width and well-quasi-ordering is very recent. Daligault, Rao and Thomassé [23] suggested that clique-width may be used as a tool towards resolving a long-standing conjecture due to Pouzet [33], which concerns the following strengthening of well-quasi-ordering. For $n \in \mathbb{N}$, a graph class is *n-well-quasi ordered* if it contains no n-coloured infinite antichain: that is, an infinite set of graphs where the vertices are coloured using n colours, and the ordering refined so that in embedding one graph as an induced subgraph of another, the colours of the vertices must match (sometimes called the *labelled induced subgraph* ordering). Pouzet's conjecture predicts that 2-well-quasi-ordering and n-well-quasi-ordering for every $n \geq 2$ are the same. Seeking an approach to prove this, Daligault, Rao and Thomassé conjecture that every 2-well-quasi-ordered graph class must have bounded clique-width. In fact, they ask whether the following stronger statement, which we give as a conjecture, is true.

Conjecture 2. *Every well-quasi-ordered graph class has bounded clique-width.*

Our second major objective is to prove this conjecture, which by the algorithmic results of [22] would add considerable motivation to finding a complete answer to Question 1.

The rest of this background section describes some of the tools available for us to use, and will focus-in on the specific problems needing resolved to answer Question 1 and prove Conjecture 2.

Infinite antichains of permutations and words

Informing our study of well-quasi-ordering for graphs are insights and results from two other combinatorial structures, which we now describe.

In the combinatorial study of permutations, the analogue of the 'induced subgraph' ordering is called *permutation containment*, and downsets in this ordering are correspondingly called *permutation classes*. By means of a direct translation, permutation classes correspond to certain types of graph class, and this translation preserves the property of being well-quasi-ordered. Conversely, an infinite antichain of permutations does not always give an infinite graph antichain, but in the important cases it does [7].

By taking a graphical perspective of permutations, it has been possible to construct large numbers of infinite antichains of permutations – three examples are shown in Figure 2. Although these antichains can be arbitrarily complex, they all exhibit a striking 'periodicity' in their construction. An earlier EPSRC grant (number EP/J006130/1) managed by the PI established the process of translating the state-of-the-art for permutations [11] to the induced subgraph ordering.

We also mention here words over a finite alphabet, equipped with the factor order (i.e. contiguous substrings). Unlike the subword order (which Higman's Theorem [28] proves is always a wqo), the factor order admits infinite antichains. However, a recent result due to the RA with Lozin and Moshkov [9], gives a decision procedure that answers the analogue of

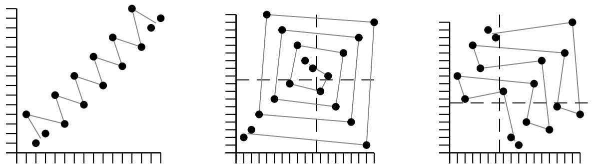

Figure 2: Three typical elements from three different infinite permutation antichains. The grey lines connecting the points indicate the periodic process by which the permutations are constructed in each case.

Question 1 for downsets in factor order. In fact, their procedure only needs to consider infinite antichains which possess a periodic construction. This, perhaps, provides the first solid evidence that well-quasi-ordering in combinatorial structures is better behaved than in arbitrary partial orders, particularly as the known infinite antichain constructions of graphs typically embed a factor order into the vertices.

The speed of a graph class

As well as directly influencing clique-width and well-quasi-orderability, the level of structure of a graph class also naturally determines the number of graphs it contains. If $\mathcal{C}_n$ denotes the set of graphs in the class $\mathcal{C}$ on the vertex set $\{1, 2, \ldots, n\}$, then the *speed* of $\mathcal{C}$ is the function $|\mathcal{C}_n|$, recording the number of graphs of each size. In their seminal paper, Scheinerman and Zito [36] identified that the asymptotic behaviour of the speed of a graph class is severely constrained, and falls into one of five different layers: from 'slowest' to 'fastest', these are constant, polynomial, exponential, factorial and superfactorial.

For a class $\mathcal{C}$ in the superfactorial layer, Allan, Lozin and Rao [19] demonstrated that $\mathcal{C}$ cannot have bounded clique-width, purely because of the sheer number of graphs that $\mathcal{C}$ must contain. Little is known about structure or well-quasi-ordering of superfactorial classes, although if true Conjecture 2 must imply every class in this layer is not well-quasi-ordered.

Balogh, Bollobàs and Morris [20] showed that the factorial layer can further be divided into two, namely classes whose speeds are dominated by (which we call *below*) or dominate (called *above*) the sequence of Bell numbers, which count the number of partitions of a set. More recently it was shown to be decidable whether a finitely defined class is above or below the Bell numbers by the RA, together with Collins, Foniok and Lozin [8], although the computational complexity of this procedure is unknown.

Below the Bell number (including the constant, polynomial and exponential layers), any class $\mathcal{C}$ has bounded clique-width (see [19]) and a structural characterisation from [30] enabled the RA to show that $\mathcal{C}$ must also be well-quasi-ordered [5].

Above the Bell number, the authors of [20] introduced a parameter called the *distinguishing number* of a graph class. Roughly speaking, this gives a measure of the number and sizes of different neighbourhoods that vertices from a graph in the class can possess, and they separate the cases where the distinguishing number of $\mathcal{C}$ is bounded (whence we say $\text{dn}(\mathcal{C}) < \infty$), from the cases where there is no bound on this number (which we call $\text{dn}(\mathcal{C}) = \infty$).

Infinite distinguishing number

For a class $\mathcal{C}$ above the Bell number with $\text{dn}(\mathcal{C}) = \infty$, Balogh, Bollobàs and Morris [20] established that $\mathcal{C}$ must contain one or more of 13 minimal classes. By means of various complementation operations, these classes can be divided into essentially three categories which we will call *Types A, B and C* – see Figure 3.

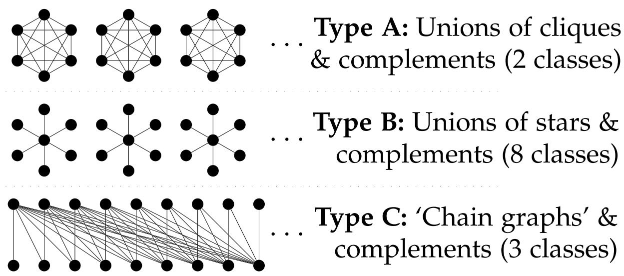

Figure 3: The 3 types of minimal classes with infinite distinguishing number.

Since these 13 minimal classes are easily described, it can readily be decided whether a finitely defined graph class $\mathcal{C}$ has $\text{dn}(\mathcal{C}) < \infty$ or $\text{dn}(\mathcal{C}) = \infty$.

Finite distinguishing number

The minimal classes with finite distinguishing number and which lie above the Bell number were recently classified by the RA, together with Collins, Foniok and Lozin [8]. All such classes belong to one infinite family (described later). This classification enabled the authors of [8] to give a decision procedure to determine whether a finitely defined class is above or below the Bell number, although there is no guarantee on how long the algorithm will take as a function of the size of the input.

In his thesis [5], the RA further demonstrated that none of this family of minimal classes is well-quasi-ordered, by exhibiting infinite antichains with periodic constructions in every case. It is also readily checked that each class in this minimal family has bounded clique-width, so for any class with finite distinguishing number (which includes all classes below the Bell number), Conjecture 2 is true, and Question 1 is effectively answerable, albeit with unknown computational complexity.

Figure 4 summarises the current state of knowledge, and the focus for this proposal.

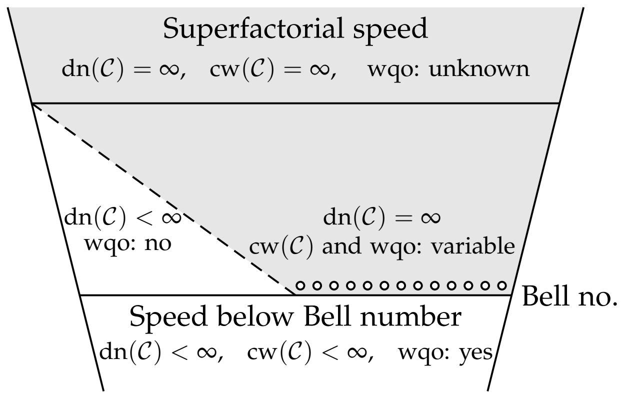

Figure 4: The relationship between the distinguishing number (dn), clique-width (cw) and well-quasi-ordering (wqo) of a graph class $\mathcal{C}$ of given speed. The shaded regions, where $\mathrm{dn}(\mathcal{C}) = \infty$, are the focus for this proposal.

National Importance and academic impact

This proposal comprises fundamental research in discrete mathematics, with clear connections to computer science. This interdisciplinary region has been identified by EPSRC as a 'priority' area, due to the leading role that the UK plays here, and the potential for impact arising from the application of mathematical theory to computational complexity. The need to enhance the foundations upon which algorithms for data sets are developed is growing increasingly important, especially with the forthcoming opening of the UK's Alan Turing Institute on the horizon.

Furthermore, this proposal involves collaboration with Vadim Lozin, who until recently was funded by grant EP/I01795X/1 specifically looking at clique-width. By building on this earlier EPSRC-funded research, the UK's reputation in this important field will be reinforced.

The transformative impact of the celebrated Graph Minor Theorem clearly illustrates the importance of concepts such as well-quasi-ordering, structural graph theory and graph parameters. Question 1 has been much-studied for many years, so the outcomes of this project are likely to attract interest from the international communities both in mathematics and in computer science. The simultaneous resolution of Conjecture 2 adds significance to the resolution of Question 1, particularly for those working in computational complexity.

To ensure researchers benefit from this proposal, significant funds have been requested in order to present results at conferences. It is also planned to invite senior researchers in the area to speak at the Open University's Winter Combinatorics Meeting, which will provide a platform to highlight this area of research, and to engage in discussions.

Two specific collaborators are identified: Vadim Lozin is a world-leader in the study of clique-width and of well-quasi-ordering, and so will bring a wealth of directly relevant graph-theoretic expertise to all aspects of the project. Vincent Vatter has a successful history of collaboration with the PI, and the intersection of research interests makes him a valuable addition to this proposal, through a small number of intensive research visits.

Research Hypothesis and Objectives

This proposal has two goals: to provide an effective decision procedure to answer Question 1, and to establish a connection between clique-width and well-quasi-ordering by resolving Conjecture 2. Underpinning both of these is a need to characterise the minimal non-well-quasi-ordered graph classes:

Objective 1. *Develop a structure theory to classify all the minimal graph classes which have infinite distinguishing number, and are not well-quasi-ordered.*

Building on this first objective, the two remaining objectives will complete the resolution of Conjecture 2 and Question 1, respectively.

Objective 2. *Establish Conjecture 2. That is, prove that every class with unbounded clique-width is not well-quasi-ordered.*

Objective 3. *Develop a decision procedure for determining whether a finitely defined graph class is well-quasi-ordered or not.*

The timeliness of this proposal follows from the recent resolution of the question of well-quasi-orderability for finite distinguishing number by the RA [5, 8]: the major steps in this proposal are only now feasible because of the techniques developed in pursuit of that result. However, we must emphasise that the work (detailed below) towards the three objectives is not merely an extension of [5, 8], and novel ideas and techniques will be needed at every stage.

Programme and Methodology

This project is divided into three work packages, one for each of the three primary objectives. Building on existing collaboration between the PI and RA, we expect both researchers to work on most parts of the project, but we have identified who will take the lead in each activity within the work packages. The overall management of the project will be overseen by the PI.

Work Package 1: *Well-quasi-ordering*

The starting point towards the resolution of Objective 1 is the classification of the minimal non-well-quasi-ordered graph classes with finite distinguishing number given by the RA in [5]. All such minimal classes belong to the same family, namely they are all so-called '$P(w,H)$ classes': here, H is a finite graph, w is an infinite periodic word on the alphabet made up from the vertices of H, and, roughly speaking, graphs in the class $P(w,H)$ correspond to finite subwords of w, with the adjacency of vertices in the graphs being determined by the structure of H.

Our first step is to investigate a generalisation of $P(w,H)$ classes, where we replace the finite graph H with a finite directed graph D, and modify the construction of graphs in the class.

Activity 1.1 (RA-led)**.** *Initiate the study of $P(w,D)$ classes, and show that they are not well-quasi-ordered.*

Crucially, the $P(w,D)$ classes can contain minimal classes of type C (that is, chain graphs and their relatives). For a class $\mathcal{C}$ that does *not* contain any $P(w,D)$ class or type C class (but can contain classes of types A or B), we believe $\mathcal{C}$ must be well-quasi-ordered.

Activity 1.2 (PI-led)**.** *Prove that any class containing no $P(w,D)$ subclass or class of type C is well-quasi-ordered.*

The prior belief underlying this activity comes from the study of permutations: the analogous result here is immediately true, arising as a consequence of a stronger property guaranteed by the PI's work on 'simple permutations' [12, 16]. Moreover, the family of $P(w,D)$ classes is believed to contain all existing infinite antichain constructions, so a counterexample would also be an extremely interesting new discovery.

The final activity in this work package, which will complete the description of the boundary of well-quasi-ordering, presents the greatest challenge in this proposal.

Activity 1.3 (PI-led)**.** *Prove that if a class contains a type C subclass but no class of the form $P(w,D)$, then it is well-quasi-ordered.*

In order to achieve this, we plan to establish a structural characterisation of the classes in question. Existing concepts and techniques from graph theory, such as 'k-uniform graphs' (used to prove the well-quasi-orderability of all classes below the Bell number), modular decomposition and lettericity, will need to be extended and combined. There are also several notions (such as those in [11]) from the study of well-quasi-ordering of permutations which can be translated into the language of graphs. This process of translation was initiated by the PI as part of EPSRC grant EP/J006130/1, see, for example [7].

Work Package 2: *Boundedness of clique-width*

This work package, whose ultimate aim is to resolve Conjecture 2, builds on the results from Work Package 1. If the $P(w,D)$ classes do indeed characterise the boundary of well-quasi-ordering, then the interface between bounded and unbounded clique-width likely lies strictly above this. We expect $P(w,D)$ classes to have bounded clique-width because of the way in which they are formed. Confirming this is our first task:

Activity 2.1 (RA-led)**.** *Show that every $P(w,D)$ class has bounded clique-width.*

If the minimal non-well-quasi-ordered classes are indeed contained in $P(w,D)$ as we believe *a priori*, then there remains one area where there are well-quasi-ordered classes which could have unbounded clique-width. Using the structural characterisation from Activity 1.2, we seek to prove the following.

Activity 2.2 (RA-led)**.** *Prove that a class of graphs which is not contained in nor contains any class of the form $P(w,D)$, and which does not contain a class of type C, has bounded clique-width.*

Work Package 3: ***Algorithms***

In parallel to the structural work in Work package 1 which is expected to characterise the minimal non-well-quasi-ordered classes, some further work is required to establish an effective decision procedure.

The first task builds on the existing algorithm in [8] to check whether a class contains some $P(w, H)$, where w is a periodic word and H is a finite graph. To limit the number of cases we check (and thus find the run time of the algorithm), we need a bound on the size of the period of w as a function of the minimal forbidden graphs in a class.

Activity 3.1 (RA-led)**.** *Establish an upper bound on the computational complexity of the algorithm to decide whether finitely defined classes with finite distinguishing number are well-quasi-ordered.*

The next task is to combine the outcome of this first task with the exploration of the classes $P(w, D)$ in Activity 1.1.

Activity 3.2 (RA-led)**.** *Establish a decision procedure with known complexity to determine whether a class contains some* $P(w, D)$.

If $P(w, D)$ does indeed define the minimal non-well-quasi-ordered classes, then Activity 3.2 should essentially complete the answer to Question 1. Taking into account the possibility that the procedure developed in Activity 3.2 leaves some aspects still unresolved or that the picture is more complicated than we believe *a priori*, we reserve the final period of this work package to 'wrap-up' and write-up.

Activity 3.3 (PI-led)**.** *Complete and write up the decision procedure for well-quasi-ordering.*

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