Permutations and permutation graphs

Robert Brignall

Schloss Dagstuhl, 8 November 2018

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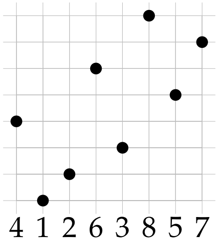

- Permutation $\pi = \pi(1) \cdots \pi(n)$
- Inversion graph G_π: for $i < j$, $ij \in E(G_\pi)$ iff $\pi(i) > \pi(j)$.
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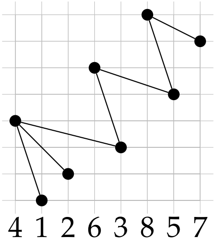

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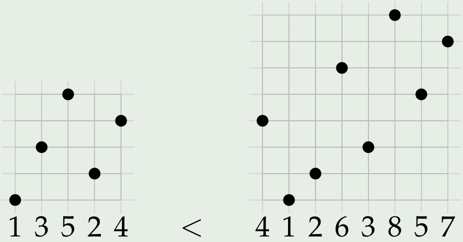
Ordering permutations: containment



- 'Classical' pattern containment: $\sigma \leq \pi$.
- Translates to induced subgraphs: $G_\sigma \leq_{\text{ind}} G_\pi$.
- Permutation class: a downset:

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$

- Avoidance: minimal forbidden permutation characterisation:

$$\mathcal{C} = \text{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$

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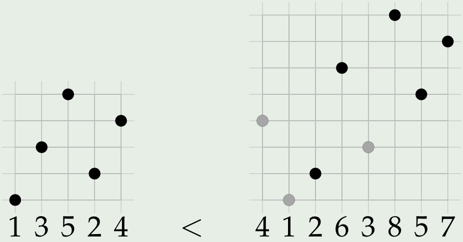

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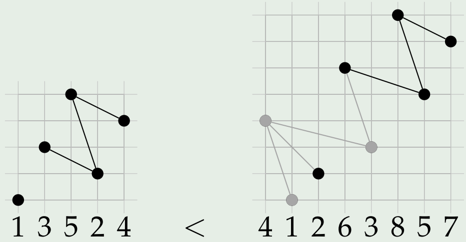

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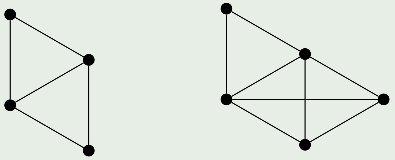

- Induced subgraph: $H \leq_{\text{ind}} G$: 'delete vertices' (& incident edges).
- Hereditary class: $\mathcal{C}$, a downset:

$$G \in \mathcal{C} \text{ and } H \leq_{\text{ind}} G \Longrightarrow H \in \mathcal{C}.$$

(Example: all planar graphs.)

- Forbidden induced subgraph characterisation, $\text{Free}(G_1, \ldots, G_k)$.

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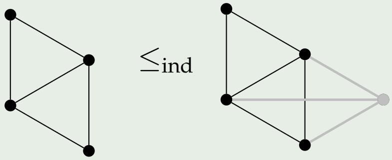

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$\text{Av}(231, 312)$	Free()
$\text{Av}(2413, 3142)$(separables)	Cographs: $\text{Free}(P_4)$
$\text{Av}(3412, 2143)$ (skew-merged)	Split permutation graphs: $\text{Free}(2K_2, C_4, C_5, S_3, \overline{\text{rising sun}}, \text{net}, \text{rising sun})$

Permutation graphs as a graph class

Graphclasses.org tells me that:

$$\text{Perm. graphs} = \text{Free}(C_{n+4}, T_2, X_2, X_3, X_{30}, X_{31}, X_{32}, X_{33}, X_{34}, X_{36}, \\ XF_1^{2n+3}, XF_2^{n+1}, XF_3^n, XF_4^n, XF_5^{2n+3}, XF_6^{2n+2}, \\ + \text{complements})$$

N.B. (e.g.) C_{n+4} are all the cycles of length ≥ 5, so this is an infinite list.

Today

Two interactions between permutations and graphs

1. Clique width
2. Labelled well-quasi-ordering

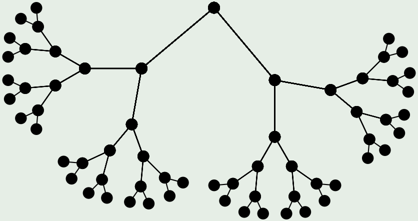
§1 Clique width

Build-a-graph

Set of labels Σ. You have 4 operations to build a labelled graph:

1. Create a new vertex with a label $i \in \Sigma$.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j, where $i, j \in \Sigma$, $i \neq j$.
4. Relabel every vertex labelled i with j.

Example (Binary trees need at most 3 labels)



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- Clique-width, $cw(G) =$ size of smallest Σ needed to build G.
- If $H \leq_{\text{ind}} G$, then $cw(H) \leq cw(G)$.
- Clique-width of a class $\mathcal{C}$

$$cw(\mathcal{C}) = \max_{G \in \mathcal{C}} cw(G)$$

if this exists.

Motivation

Theorem (Courcelle, Makowsky and Rotics (2000))

If $cw(\mathcal{C}) < \infty$, then any property expressible in monadic second-order (MSO_1) logic can be determined in polynomial time for $\mathcal{C}$.

- MSO_1 includes many NP-hard algorithms: e.g. k-colouring ($k \geq 3$), graph connectivity, maximum independent set,…
- Generalises treewidth, critical to the proof of the Graph Minor Theorem (see next slide)
- Unlike treewidth, clique-width can cope with dense graphs

Diversion: treewidth, $tw(G)$

- $tw(G)$ measures 'how like a tree' G is ($tw(G) = 1$ iff G is a tree).
- Bounded treewidth $\implies$ all problems in MSO_2 in polynomial time.

Theorem (Robertson and Seymour, 1986)

For a minor-closed family of graphs $\mathcal{C}$, $tw(\mathcal{C})$ bounded if and only if $\mathcal{C}$ does not contain all planar graphs.

- Planar graphs are the unique "minimal" family for treewidth.

Question

Can we get a similar theorem for clique width?

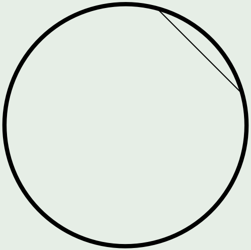
Yes! For vertex-minors. . .

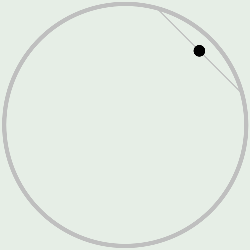
- Vertex-minor = induced subgraph + local complements

Theorem (Geelen, Kwon, McCarty, Wollan (announced 2018))

A vertex-minor-closed downset of graphs has unbounded clique-width if and only if it contains every circle graph as a vertex-minor.

Circle graph = intersection graph of chords





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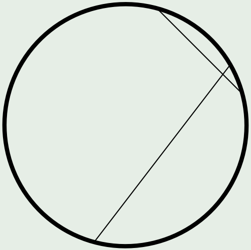

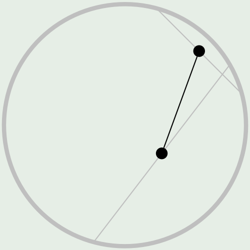

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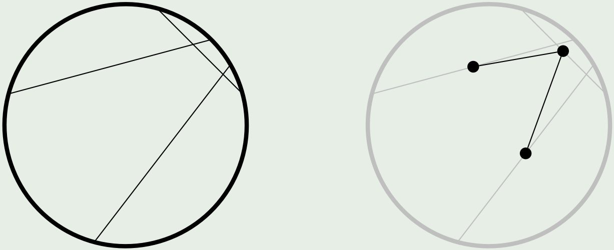

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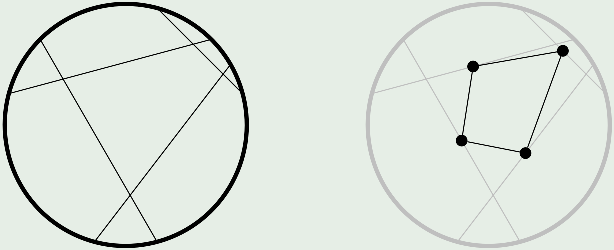

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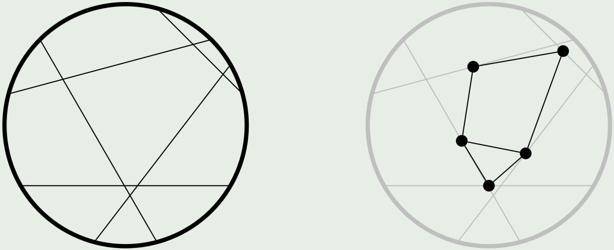

Clique width on graph classes

Bounded clique-width

- Cographs, $\mathcal{C} = \text{Free}(P_4)$: $cw(\mathcal{C}) = 2$.
- $\mathcal{F} = \{\text{forests}\}$: $cw(\mathcal{F}) = 3$.

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Unbounded clique-width

- Circle graphs
- Split permutation graphs
- Bipartite permutation graphs
- Any class with superfactorial speed
  ($\sim$ more than n^{cn} labelled graphs of order n, for any c)

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- Bipartite permutation graphs $\longleftarrow$ minimal!
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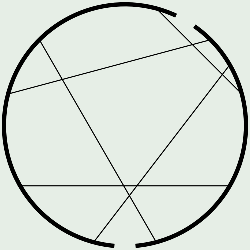
Circle graphs are minimal with respect to vertex minor, but *not* for induced subgraphs.

Permutation graphs ⊂ Circle graphs



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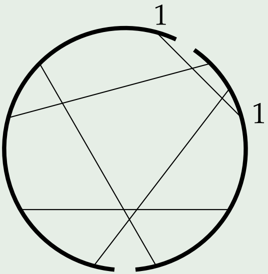

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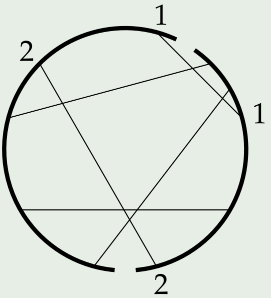

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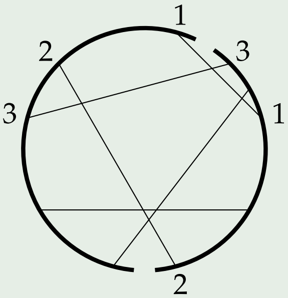

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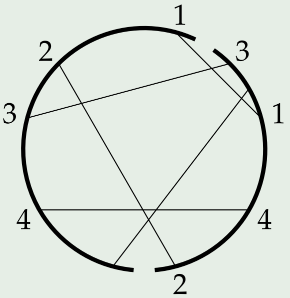

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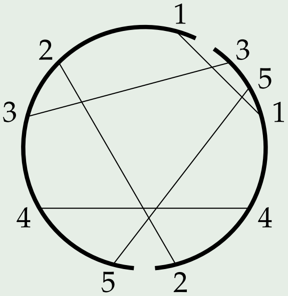

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Av(321) vs Bipartite permutation graphs

Theorem (Lozin, 2011)

Bipartite permutation graphs are a minimal class with unbounded clique-width.

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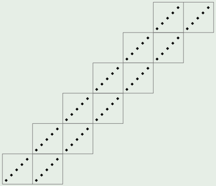
Structure:		

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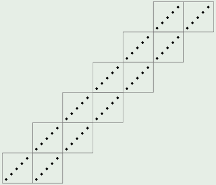
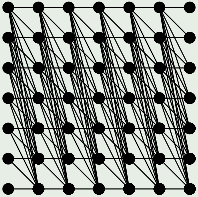
Structure:		

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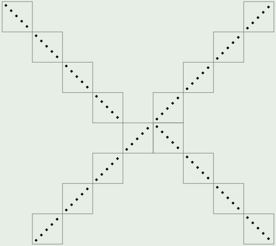
Structure:		

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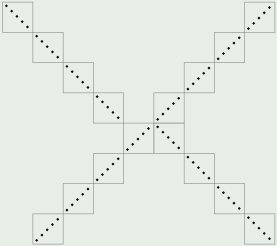
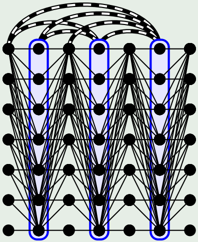
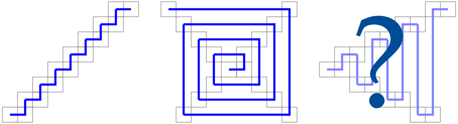
Structure:		

More minimal classes?

- Permutation class structure is a long ‘path’:



- Could find minimal classes of permutation graphs.
- Carry out local complementation to make other (non-permutation) graph classes.

Corollary (to Geelen, Kwon, McCarty, Wollan)

Every minimal class of unbounded clique width is a subclass of circle graphs.

Question (possibly naive)

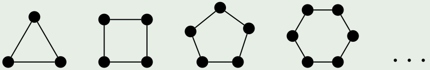
Are all these classes related to each other by local complementation?

§2 Labelled well-quasi-ordering

Infinite labelled antichains

- Antichain: set of pairwise incomparable graphs/permutations

The set of cycles forms an infinite antichain



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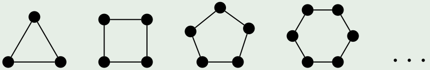

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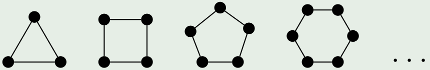

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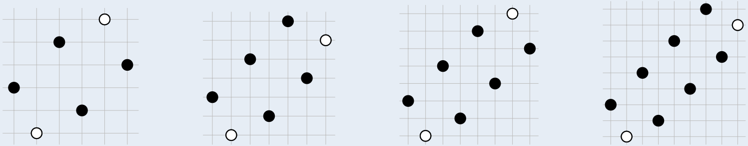
Increasing oscillations/Gollan permutations too...



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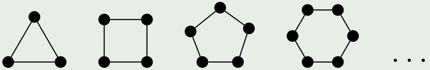

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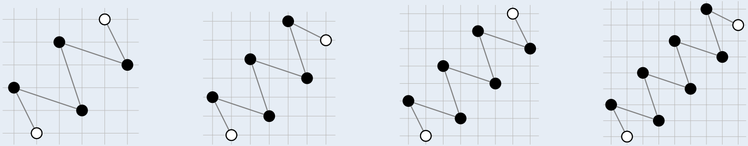

No labelled antichains

- well-quasi-order (WQO): no infinite antichain.
- Labelled well-quasi-order (LWQO): no infinite labelled antichain.

Theorem (Pouzet, 1972)

Every LWQO class (of graphs, permutations, anything) is finitely based.

Conjecture (Korpelainen, Lozin & Razgon, 2013; Atminas & Lozin, 2015)

Every finitely based WQO graph class must also be LWQO.

Intuition

Conjecture (KLR, 2013; AL, 2015)

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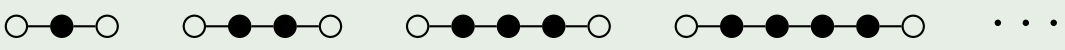
If a graph contains long paths, then it contains



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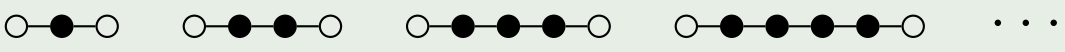

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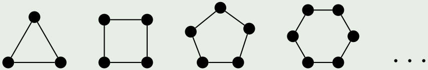
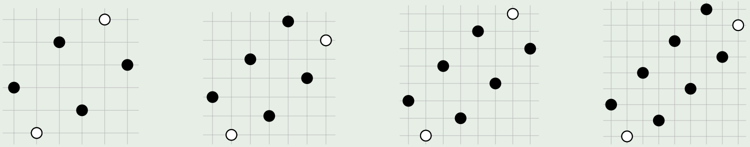
But then, you can't avoid



. . . unless they are all in the basis.

‘Obviously’ wrong for permutations

Increasing oscillations again



Proposition

The smallest class containing the increasing oscillations is $Av(321, 2341, 3412, 4123)$ *and is WQO (but not LWQO).*

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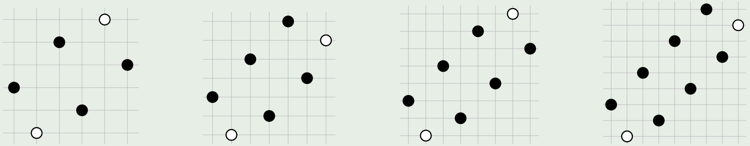

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But...

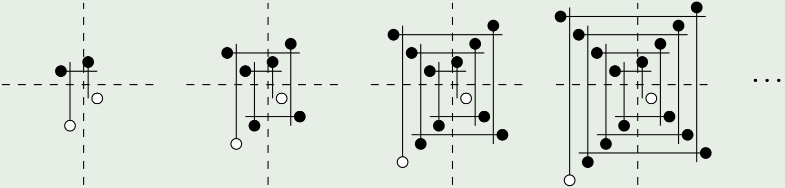
As a graph class, C_n is a basis element for $n \geq 5$.
$\Rightarrow$ not a counterexample.

Another permutation example

Proposition (B., Engen, Vatter, 2018+)

Av(2143, 2413, 3412, 314562, 412563, 415632, 431562, 512364, 512643, 516432, 541263, 541632, 543162) *is another WQO-but-not-LWQO class.*

Here's the labelled antichain



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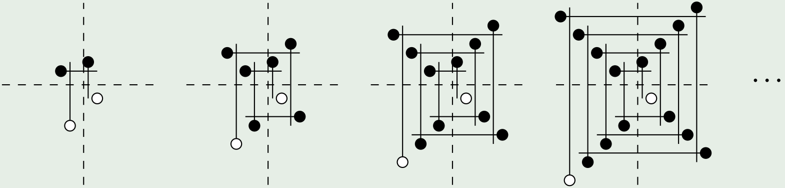


Corollary

The class Free($\underline{2K_2}$, C_4, C_5, *net*, *co-net*, *rising sun*, *co-rising sun*, H, $\overline{H}$, *cross*, *co-cross*, X_{168}, $\overline{X_{168}}$, X_{160}), *is WQO but not LWQO.*

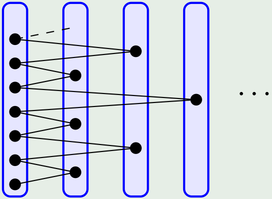
Closing thoughts: WQO and clique-width

Conjecture (Daligault, Rao, Thomassé, 2010)

If $\mathcal{C}$ is labelled well-quasi-ordered, then $\mathcal{C}$ has bounded clique-width.

N.B.

WQO does *not* imply bounded clique width (Lozin, Razgon, Zamaraev, 2018).



Thanks!