Labelled well-quasi-order for permutation classes

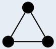
Robert Brignall

Joint work with Vince Vatter (U. Florida)

Combinatorics Seminar, University of Florida, 9 February 2021

Embedding graphs

Question (easy)

Can I delete vertices (and incident edges) from  *to produce*  *?*

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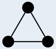
Can I delete vertices (and incident edges) from  *to produce*  *?*

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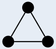
Can I delete vertices (and incident edges) from  *to produce*  *?*

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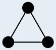
Can I delete vertices (and incident edges) from  *to produce*  *?*

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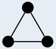
Can I delete vertices (and incident edges) from  *to produce*  *?*

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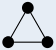
Can I delete vertices (and incident edges) from  *to produce*  *?*

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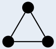
Can I delete vertices (and incident edges) from  *to produce*  *?*

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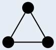
Can I delete vertices (and incident edges) from  *to produce*  *?*

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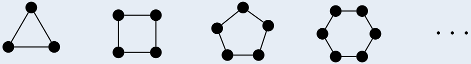
Question (slightly harder)

Is any graph in the following (infinite) list an induced subgraph of another?



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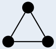
Can I delete vertices (and incident edges) from  *to produce*  *?*

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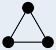
Can I delete vertices (and incident edges) from  *to produce*  *?*

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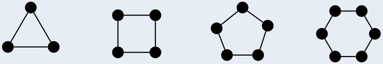 . . .

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Labelled embeddings

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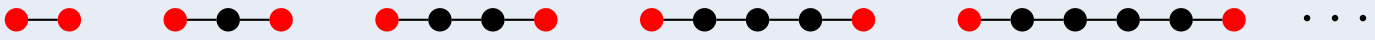

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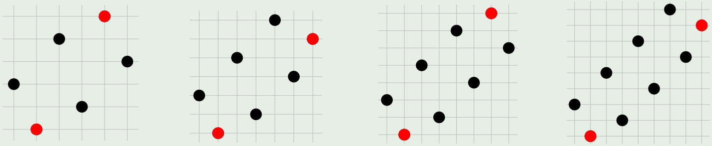
A similar phenomenon in permutations



3 1 5 2 6 4   3 1 5 2 7 4 6   3 1 5 2 7 4 8 6   3 1 5 2 7 4 9 6 8

To embed one of these as a pattern in another, we must embed a red point in a black one.

§1 Combinatorial structures

Induced substructures

Graph $G = (V, E)$

- Vertices V
- Relationship between pairs of vertices: Edges $u \sim v$ for $u, v \in V$.

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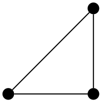

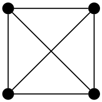

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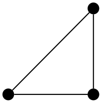
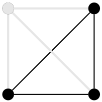
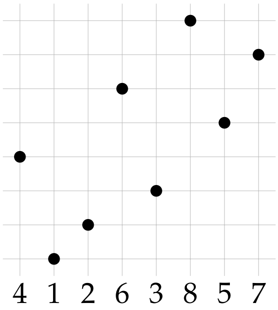
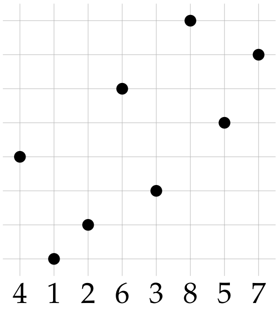


$\leq_{\text{ind}}$




4 1 2 6 3 8 5 7

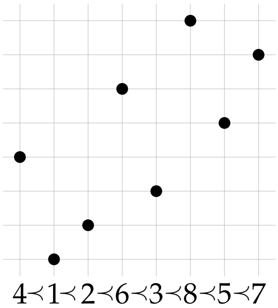


Permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$

- 'Vertices': $V = \{1, 2, \ldots, n\}$
- Relationship between pairs of 'vertices': given by two linear orders, $<$ and $\prec$

$$1 < 2 < \cdots < n$$
$$\pi(1) \prec \pi(2) \prec \cdots \prec \pi(n)$$

($\prec$ is the 'reading order' of the permutation)



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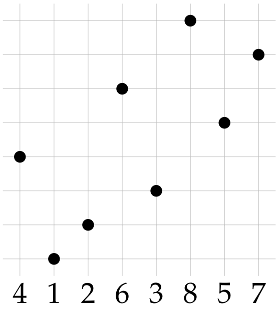
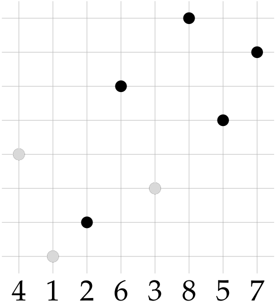
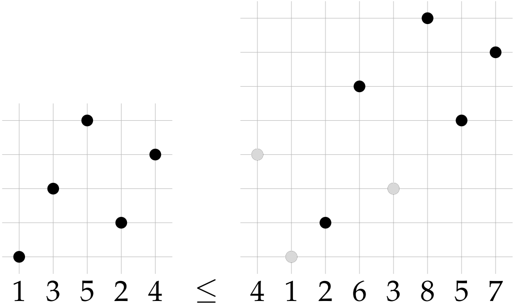


- Induced subpermutation ordering: containment



- Induced subpermutation ordering: containment
- ‘Delete entries, and rescale’



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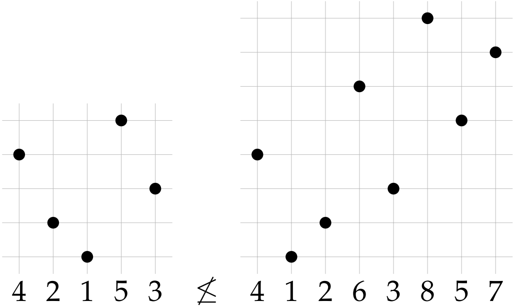

- Induced subpermutation ordering: containment
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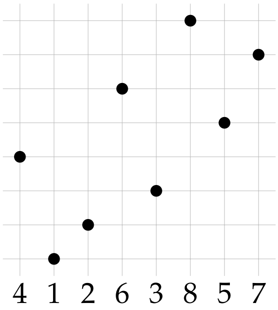
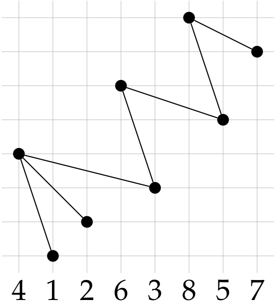
- If $\sigma \not\leq \tau$, then τ avoids σ.



Inversion graph G_π of $\pi = \pi(1) \cdots \pi(n)$:

- Vertices $= \{1, 2, \ldots, n\}$
- Edges: $a \sim b$ if $a < b$ and $b \prec a$ (edges $=$ inversions)



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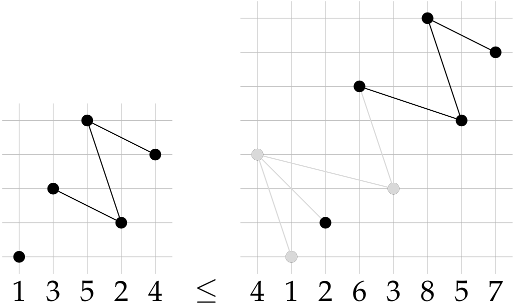

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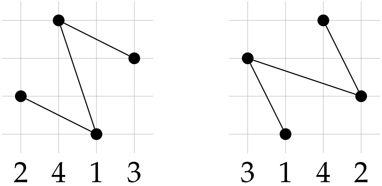
Induced substructure preserved: $\sigma \leq \pi$ implies $G_\sigma \leq_{\text{ind}} G_\pi$

Permutations to graphs is many-to-one

$\sigma \leq \pi$ implies $G_\sigma \leq_{\text{ind}} G_\pi$ but:



$G_{2413} \cong G_{3142} \cong$  even though $2413 \neq 3142$.

In general, there exist arbitrarily large sets of permutations with the same inversion graph.

§2 Hereditary classes and WQO

Hereditary classes

Set $\mathcal{C}$ of graphs/permutations is hereditary if
$A \in \mathcal{C}$ and B is an induced substructure of A, then $B \in \mathcal{C}$. ('class')

Every hereditary class has a unique set of minimal forbidden elements:
the smallest things that are 'not in the class'. ('basis')

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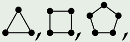
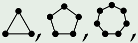
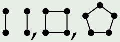
Some graph classes

Class $\mathcal{C} = \mathrm{Free}(\mathfrak{B})$	Basis $\mathfrak{B}$
Empty graphs (no edges)	$\{$  $\}$
Forests	$\{$  $, \ldots \}$
Bipartite graphs	$\{$  $, \ldots \}$
Split (clique $+$ independent)	$\{$  $\}$
Inversion graphs	$\mathrm{Free}(C_{n+4}, T_2, X_2, X_3, X_{30}, X_{31}, X_{32}, X_{33}, X_{34}, X_{36}, XF_1^{2n+3}, XF_2^{n+1}, XF_3^{n}, XF_4^{n}, XF_5^{2n+3}, XF_6^{2n+2}, +\text{complements})$ (Gallai 1967)

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Some permutation classes

Class $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$	Basis $\mathfrak{B}$
$\{1, 12, 123, \dots\}$	$\{21\}$
Union of 2 increases	$\{321\}$
Union of increase & decrease	$\{3412, 2143\}$
'Stack sortable'	$\{231\}$
'2-stack-sortable'	Infinite (Murphy 2003)

Note that…

…no one basis element embeds in another.

They are antichains.

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They are antichains.

…forests, bipartite graphs, inversion graphs and 2-stack sortable permutations have an infinite basis.

They are infinite antichains.

Infinite antichains in two places

The basis is an antichain

A class can be finitely or infinitely based.

Antichains inside the class

If a class doesn't *contain* an infinite antichain, it is well-quasi-ordered (WQO).

Motivation: tame vs wild

Finitely based classes

Structures in the class tend to be 'nice'
Can use 'basis' as input for algorithms.

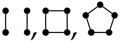
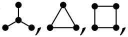
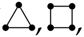
WQO classes

Structures in the class tend to be 'nice'
Only countably many subclasses

Finitely based WQO classes

All of the above, plus:
Every subclass is finitely based

Graph classes

	WQO	**Not WQO**
Finitely based	Cographs Free()	Split graphs Free()
Infinitely based	Linear forests Free(, . . .)	Forests Free(, . . .)

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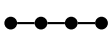
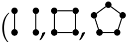
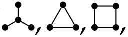
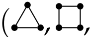
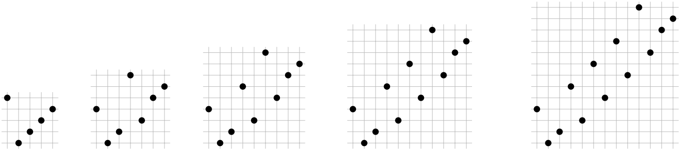
Finitely based	Cographs Free()	Split graphs Free()
Infinitely based	Linear forests Free(, . . .)	Forests Free(, . . .)

Permutation classes

	WQO	**Not WQO**
Finitely based	Separables $\mathrm{Av}(2413, 3142)$	Increase $\cup$ decrease $\mathrm{Av}(2143, 3412)$
Infinitely based	$\mathrm{Av}(321, 3412, 2341,$ $251364, \mathfrak{Osc})$	$\mathrm{Av}(\mathfrak{Osc})$

Where $\mathfrak{Osc}$ is:



5 1 2 3 4   4 1 2 7 3 5 6   4 1 2 6 3 9 5 7 8   4 1 2 6 3 8 5 11 7 9 10   4 1 2 6 3 8 5 10 7 13 9 11 12

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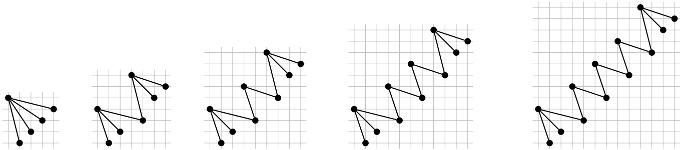
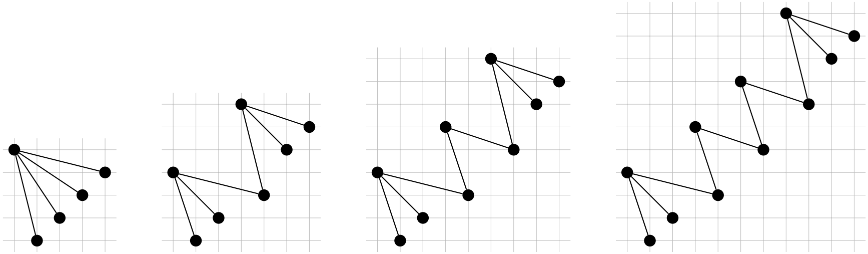


§3 Labelled WQO

A regular infinite antichain:



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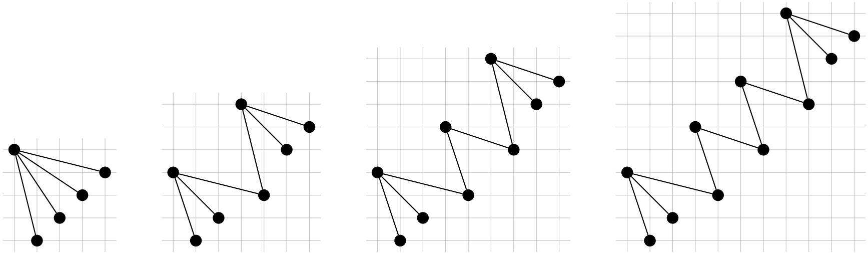

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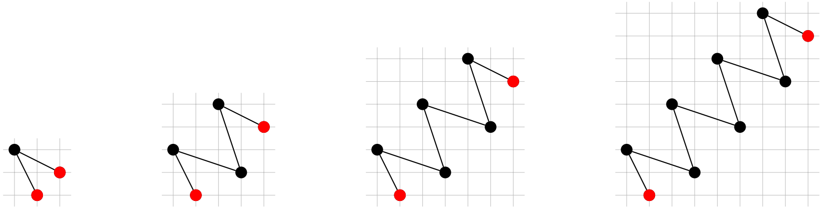


Labels can be (partially) ordered (e.g. $\bullet \preceq \bullet$): embedding must respect the label ordering.

Labelled WQO

A class is labelled well-quasi-ordered (LWQO) if we cannot construct a labelled infinite antichain, no matter the set of labels.[†]

[†] Includes infinite sets of labels, but they must be WQO.

	LWQO	**Not LWQO**
Finitely based	Separables $\mathrm{Av}(2413, 3142)$	Union of 2 increases $\mathrm{Av}(321)$
Infinitely based	None	$\mathrm{Av}(321, 3412, 2341, 251364, \mathfrak{Osc})$

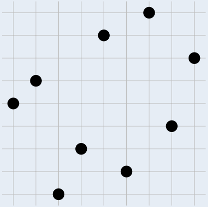
No labelled antichain $\Rightarrow$ finite basis

Proposition (After Pouzet, 1972)

An LWQO (permutation) class $\mathcal{C}$ is finitely based.

Proof.

Write $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$. For each $\beta \in \mathfrak{B}$:



β

$\square$

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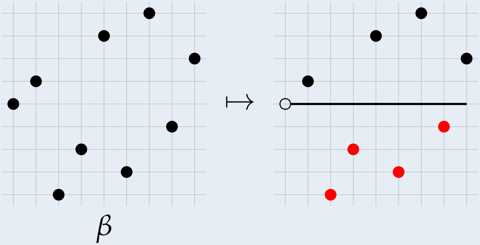

□

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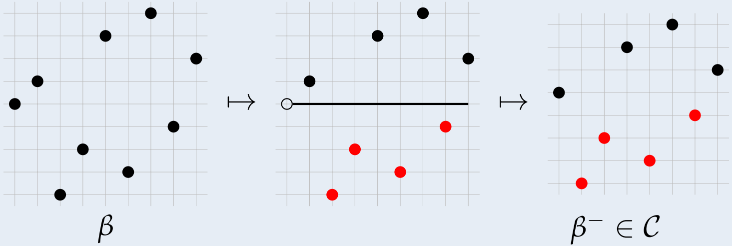


$\mathfrak{B}^- = \{\beta^- : \beta \in \mathfrak{B}\}$ is a labelled antichain in $\mathcal{C}$: must be finite. $\square$

LWQO vs (WQO + finite basis)

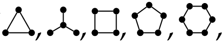
Conjecture (Korpelainen, Lozin & Razgon, 2013)

Every finitely based WQO graph class is LWQO.

Not true for permutations:

Proposition

The class $\mathcal{C} = Av(321, 2341, 3412, 4123)$ *is WQO but not LWQO.*

But $G_{\mathcal{C}} = \text{Free}($  $, \ldots)$ is not finitely based, so this is not a counterexample to the conjecture.

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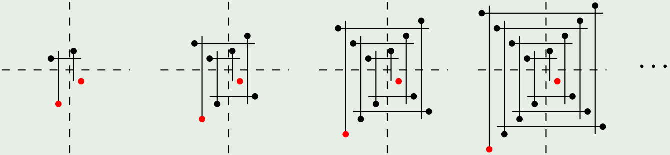
Every finitely based WQO graph class is LWQO.

Proposition (B., Engen, Vatter, 2018)

$\mathcal{D} = Av(2143, 2413, 3412, 314562, 412563, 415632, 431562, 512364, 512643, 516432, 541263, 541632, 543162)$ *is WQO but not LWQO.*

Here's the labelled antichain



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Corollary

The class $G_{\mathcal{D}} = Free($, , , *net, co-net, rising sun, co-rising sun,* H, $\overline{H}$, *cross, co-cross,* X_{168}, $\overline{X_{168}}$, X_{160}), *is WQO but not LWQO.*

…so the conjecture is false. LWQO is *strictly* stronger than WQO + finite basis.

One-point extensions

$\mathcal{C}^{+1} = \{\pi : \text{some entry of } \pi \text{ can be removed to form } \pi^- \in \mathcal{C}\}.$

If $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$, then $\mathfrak{B} \subseteq \mathcal{C}^{+1}$, thus:

Observation

$\mathcal{C}^{+1}$ WQO implies $\mathcal{C}$ is WQO and finitely based.

Conversely…

Lemma (Atkinson and Beals, 1999)

If $\mathcal{C}$ is finitely based, then $\mathcal{C}^{+1}$ is finitely based.

But: $\mathcal{C}$ WQO does not imply $\mathcal{C}^{+1}$ WQO (regardless of basis).

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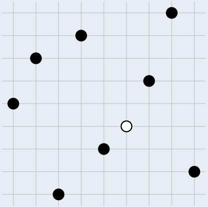
Proposition (B., Vatter)

$\mathcal{C}$ is LWQO if and only if $\mathcal{C}^{+1}$ is LWQO.

Main proof idea, showing $\mathcal{C}$ LWQO $\Rightarrow$ $\mathcal{C}^{+1}$ WQO.

For $\pi \in \mathcal{C}^{+1}$, encode 'extra' point using 4 labels on π^-:



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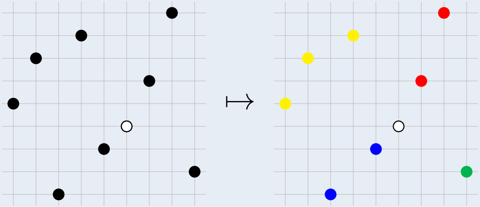

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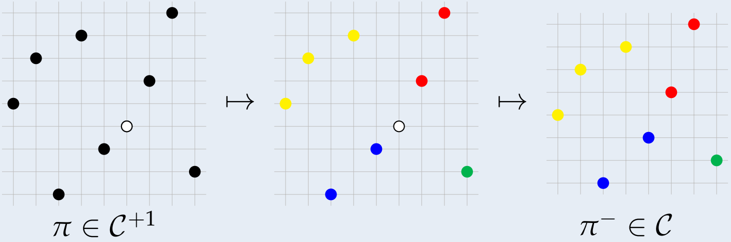

So: any antichain in $\mathcal{C}^{+1}$ corresponds to a labelled antichain in $\mathcal{C}$. $\square$

§4 Permutations & inversion graphs

Does WQO translate?

Recall: $\sigma \leq \pi \Rightarrow G_\sigma \leq_{\text{ind}} G_\pi$. Thus $\mathcal{C}$ (L)WQO $\Rightarrow G_{\mathcal{C}}$ (L)WQO.

Question

If $\mathcal{C}$ is a permutation class such that $G_{\mathcal{C}}$ is WQO, must $\mathcal{C}$ be WQO?

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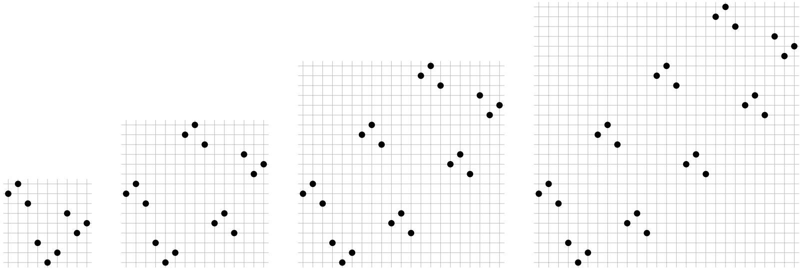

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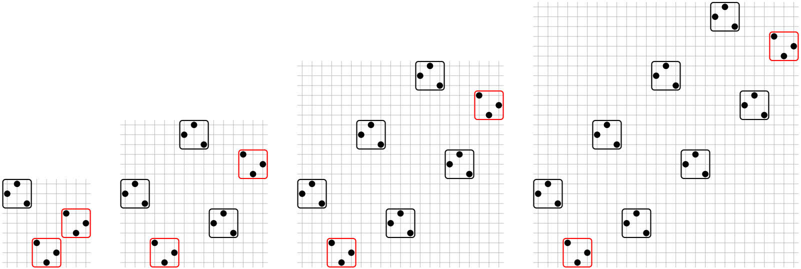

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The proof needs several ingredients:

- The substitution decomposition (a.k.a. modular decomposition)
- Nash-Williams' 1963 minimal bad sequence argument (needs Axiom of Dependent Choice)
- Gallai's 1967 characterization of the 'preimages' of G_π,

$$\Sigma_\pi = \{\text{permutations } \sigma : G_\sigma \cong G_\pi\}.$$

  We restrict to simple permutations, in which case $|\Sigma_\pi| \leq 4$.
- A 2019 result of Klavík and Zeman concerning automorphism groups of prime inversion graphs.

Some open questions and conjectures

Conjecture

If the permutation class $\mathcal{C}^{+1}$ is WQO, then $\mathcal{C}$ (and thus also $\mathcal{C}^{+1}$) is LWQO.

n-WQO: WQO when using a set of n incomparable labels.

Conjecture (Pouzet 1972)

A class of graphs is 2-WQO if and only if it is n-WQO for every $n \geq 2$.

Question

Is every 2-WQO permutation class also LWQO?

Thanks!