Labelled well-quasi-order for permutation classes

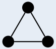
Robert Brignall

Joint work with Vince Vatter (U. Florida)

25th Ontario Combinatorics Workshop, Queen's University, 5 June 2021

Embedding graphs

Question (easy)

Can I delete vertices (and incident edges) from  *to produce*  *?*

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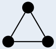
Can I delete vertices (and incident edges) from  *to produce*  *?*

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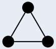
Can I delete vertices (and incident edges) from  *to produce*  *?*

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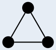
Can I delete vertices (and incident edges) from  *to produce*  *?*

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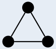
Can I delete vertices (and incident edges) from  *to produce*  *?*

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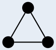
Can I delete vertices (and incident edges) from  *to produce*  *?*

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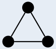
Can I delete vertices (and incident edges) from  *to produce*  *?*

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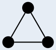
Can I delete vertices (and incident edges) from  *to produce*  *?*

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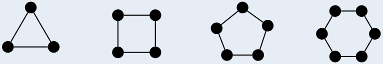
Is any graph in the following (infinite) list an induced subgraph of another?

 . . .

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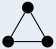
Can I delete vertices (and incident edges) from  *to produce*  *?*

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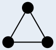
Can I delete vertices (and incident edges) from  *to produce*  *?*

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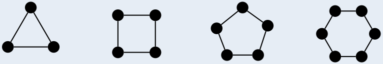 . . .

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Labelled embeddings

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Does any graph in the following (infinite) list embed as an induced subgraph of another so that the vertex colours match up?

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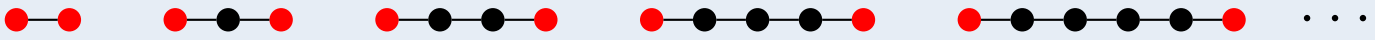

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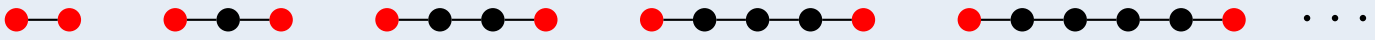

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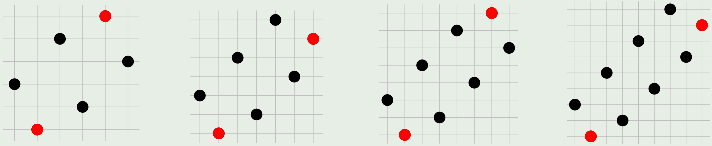
A similar phenomenon in permutations



3 1 5 2 6 4   3 1 5 2 7 4 6   3 1 5 2 7 4 8 6   3 1 5 2 7 4 9 6 8

To embed one of these as a pattern in another, we must embed a red point in a black one.

§1 Combinatorial structures

Induced substructures

Graph $G = (V, E)$

- Vertices V
- Relationship between pairs of vertices: Edges $u \sim v$ for $u, v \in V$.

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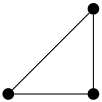

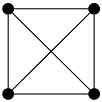

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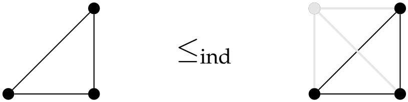
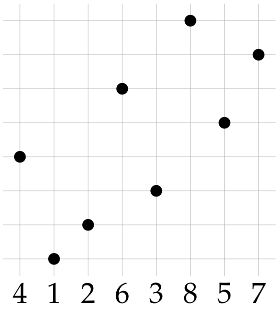
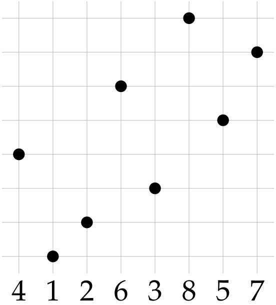



4 1 2 6 3 8 5 7

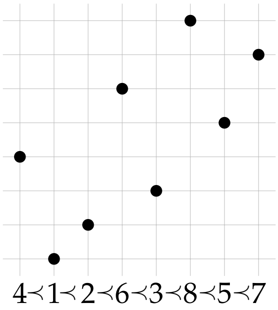


Permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$

- 'Vertices': $V = \{1, 2, \ldots, n\}$
- Relationship between pairs of 'vertices': given by two linear orders, $<$ and $\prec$

$$1 < 2 < \cdots < n$$
$$\pi(1) \prec \pi(2) \prec \cdots \prec \pi(n)$$

($\prec$ is the 'reading order' of the permutation)



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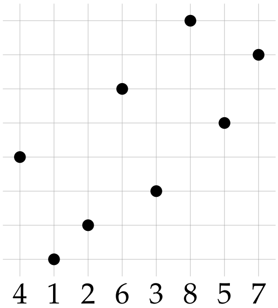
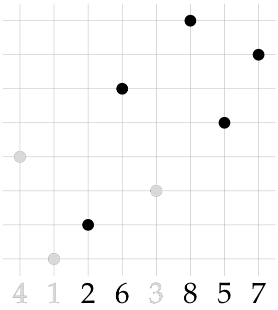
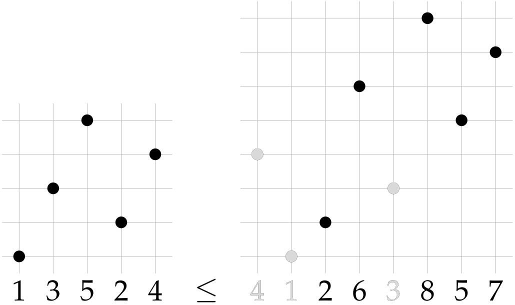


- Induced subpermutation ordering: containment



- Induced subpermutation ordering: containment
- ‘Delete entries, and rescale’



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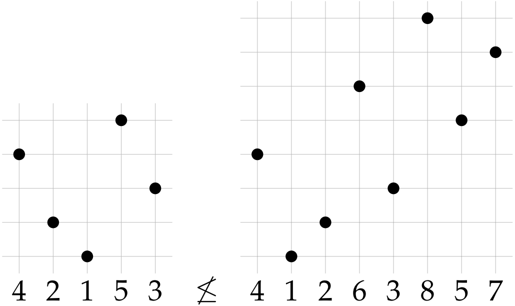
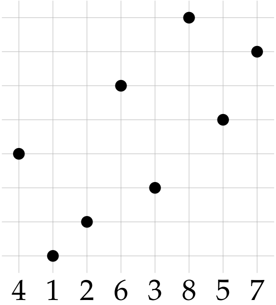
- Formally: $\sigma \leq \tau$ if τ has a subsequence with the same relative ordering as σ.

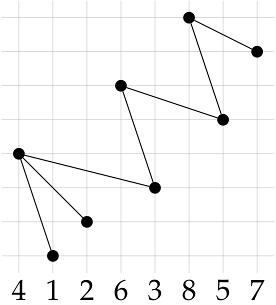


- Induced subpermutation ordering: containment
- 'Delete entries, and rescale'
- Formally: $\sigma \leq \tau$ if τ has a subsequence with the same relative ordering as σ.
- If $\sigma \not\leq \tau$, then τ avoids σ.



Inversion graph G_π of $\pi = \pi(1) \cdots \pi(n)$:

- Vertices $= \{1, 2, \ldots, n\}$
- Edges: $a \sim b$ if $a < b$ and $b \prec a$ (edges = inversions)



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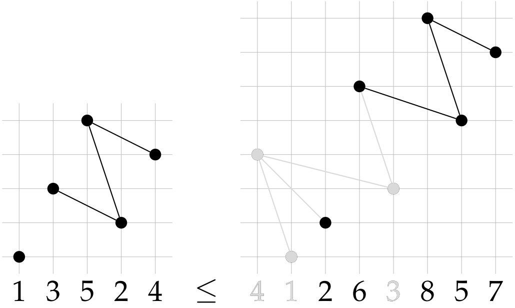

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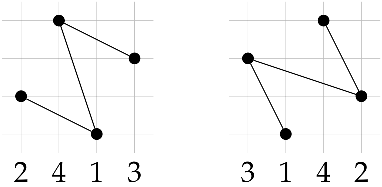
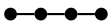
Induced substructure preserved: $\sigma \leq \pi$ implies $G_\sigma \leq_{\text{ind}} G_\pi$

Permutations to graphs is many-to-one

$\sigma \leq \pi$ implies $G_\sigma \leq_{\text{ind}} G_\pi$ but:



$G_{2413} \cong G_{3142} \cong$  even though $2413 \neq 3142$.

In general, there exist arbitrarily large sets of permutations with the same inversion graph.

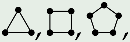
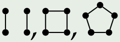
§2 Hereditary classes and WQO

Hereditary classes

Set $\mathcal{C}$ of graphs/permutations is hereditary if
$A \in \mathcal{C}$ and B is an induced substructure of A, then $B \in \mathcal{C}$. ('class')

Every hereditary class has a unique set of minimal forbidden elements:
the smallest things that are 'not in the class'. ('basis')

Some graph classes

Class $\mathcal{C} = \text{Free}(\mathfrak{B})$	Basis $\mathfrak{B}$
Empty graphs (no edges)	{  }
Forests	{ , ... }
Split (clique + independent)	{  }
Inversion graphs	$\text{Free}(C_{n+4}, T_2, X_2, X_3, X_{30}, X_{31}, X_{32}, X_{33}, X_{34}, X_{36}, XF_1^{2n+3}, XF_2^{n+1}, XF_3^n, XF_4^n, XF_5^{2n+3}, XF_6^{2n+2}, +\text{complements})$ (Gallai 1967)

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Some permutation classes

Class $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$	Basis $\mathfrak{B}$
$\{1, 12, 123, \dots\}$	$\{21\}$
Union of increase & decrease ('X')	$\{3412, 2143\}$
'Stack sortable'	$\{231\}$
'2-stack-sortable'	Infinite (Murphy 2003)

Note that…

…no one basis element embeds in another.

They are antichains.

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…no one basis element embeds in another.

They are antichains.

…forests, inversion graphs and 2-stack sortable permutations have an infinite basis.

They are infinite antichains.

Infinite antichains in two places

The basis is an antichain

A class can be finitely or infinitely based.

Antichains inside the class

If a class doesn't *contain* an infinite antichain, it is well-quasi-ordered (WQO).

Motivation: tame vs wild

Finitely based classes

Structures in the class tend to be 'nice'
Can use 'basis' as input for algorithms.

WQO classes

Structures in the class tend to be 'nice'
Only countably many subclasses

Finitely based WQO classes

All of the above, plus:
Every subclass is finitely based

Diversion: Graph Minor Theorem

Robertson and Seymour's Graph Minor Theorem says there are no infinite antichains in the graph minor ordering.
Thus, every minor-closed class is 'finitely based' and WQO.

Graph classes

	WQO	**Not WQO**
Finitely based	Cographs	Split graphs
	Free()	Free(,,)
Infinitely based	Linear forests	Forests
	Free(,,,...)	Free(,,...)

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Finitely based	Cographs Free()	Split graphs Free(, , )
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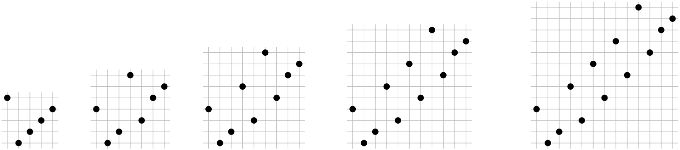
Permutation classes

	WQO	**Not WQO**
Finitely based	Separables Av(2413, 3142)	Increase ∪ decrease Av(2143, 3412)
Infinitely based	Av(321, 3412, 2341, 251364, $\mathfrak{Osc}$)	Av($\mathfrak{Osc}$)

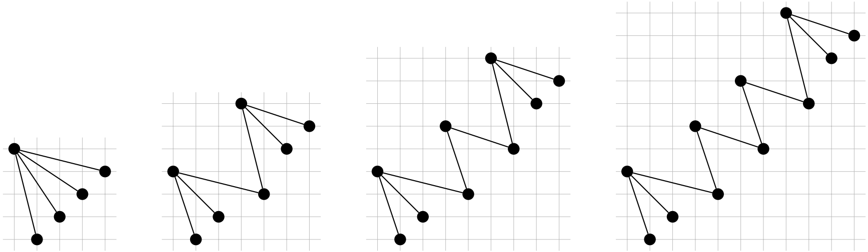
Where $\mathfrak{Osc}$ is:



5 1 2 3 4   4 1 2 7 3 5 6   4 1 2 6 3 9 5 7 8   4 1 2 6 3 8 5 11 7 9 10   4 1 2 6 3 8 5 10 7 13 9 11 12

§3 Labelled WQO

A regular infinite antichain (of permutations and/or graphs):



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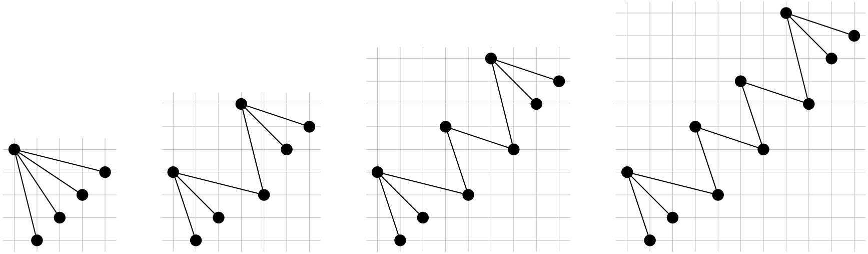
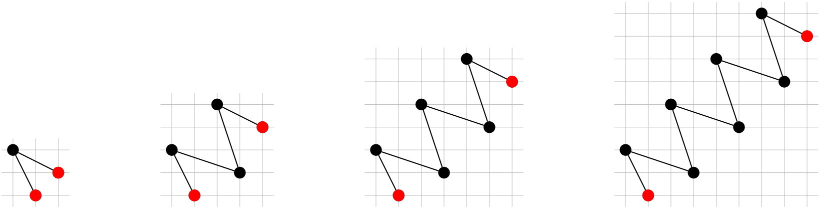


A labelled infinite antichain:



Labels can be (partially) ordered: the above is an antichain if $\bullet$ and $\bullet$ are incomparable, or if $\bullet \prec \bullet$.

Not an antichain if $\bullet \preceq \bullet$.

Labelled WQO

A class is labelled well-quasi-ordered (LWQO) if we cannot construct a labelled infinite antichain, no matter the set of labels.[†]

[†] Includes infinite sets of labels, but they must be WQO.

	LWQO	**Not LWQO**
Finitely based	Separables $\mathrm{Av}(2413, 3142)$	Increase $\cup$ decrease $\mathrm{Av}(2143, 3412)$
Infinitely based	None	$\mathrm{Av}(321, 3412, 2341, 251364, \mathfrak{Osc})$

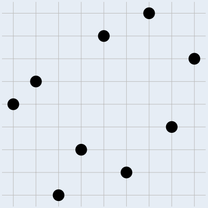
No labelled antichain $\Rightarrow$ finite basis

Proposition (After Pouzet, 1972)

An LWQO (permutation) class $\mathcal{C}$ is finitely based.

Proof.

Write $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$. For each $\beta \in \mathfrak{B}$:



β

$\square$

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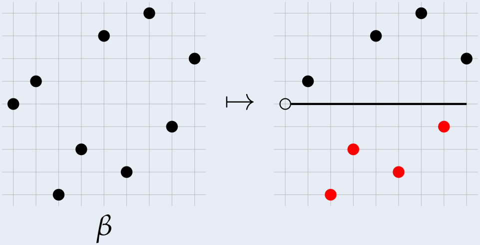

□

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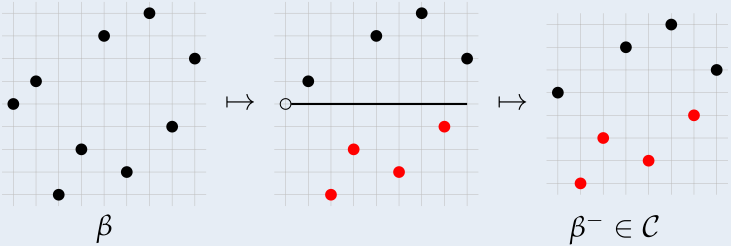


$\mathfrak{B}^- = \{\beta^- : \beta \in \mathfrak{B}\}$ is a labelled antichain in $\mathcal{C}$: must be finite. □

LWQO vs (WQO + finite basis)

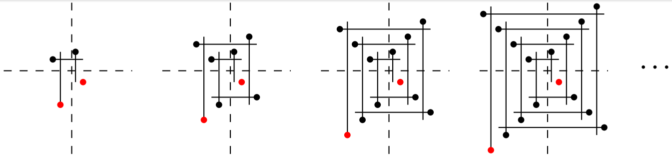
LWQO is strictly stronger than WQO + finite basis for permutations...

Proposition

The class $\mathcal{C} = Av(321, 2341, 3412, 4123)$ *is WQO but not LWQO.*

Proposition (B., Engen, Vatter, 2018)

$\mathcal{D} = Av(2143, 2413, 3412, 314562, 412563, 415632, 431562, 512364, 512643, 516432, 541263, 541632, 543162)$ *is WQO but not LWQO.*



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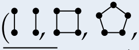
…and for graphs…

Corollary

The class $G_{\mathcal{D}} = Free($  *, net, co-net, rising sun, co-rising sun,* H, $\overline{H}$, *cross, co-cross,* X_{168}, $\overline{X_{168}}$, $X_{160})$, *is WQO but not LWQO.*

One-point extensions

$\mathcal{C}^{+1} = \{\pi : \text{some entry of } \pi \text{ can be removed to form } \pi^- \in \mathcal{C}\}$.

Proposition (B., Vatter)

$\mathcal{C}$ is LWQO if and only if $\mathcal{C}^{+1}$ is LWQO.

Note: $\mathcal{C}$ WQO does not imply $\mathcal{C}^{+1}$ WQO.

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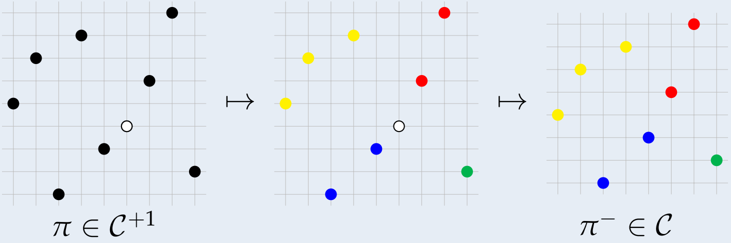
Sketch proof of $\mathcal{C}$ LWQO $\Rightarrow \mathcal{C}^{+1}$ WQO.

Any antichain in $\mathcal{C}^{+1}$ corresponds to a 4-labeled antichain in $\mathcal{C}$:



□

Note: $\mathcal{C}$ WQO does not imply $\mathcal{C}^{+1}$ WQO.

§4 Permutations & inversion graphs

Does WQO translate?

Recall: $\sigma \leq \pi \Rightarrow G_\sigma \leq_{\text{ind}} G_\pi$.

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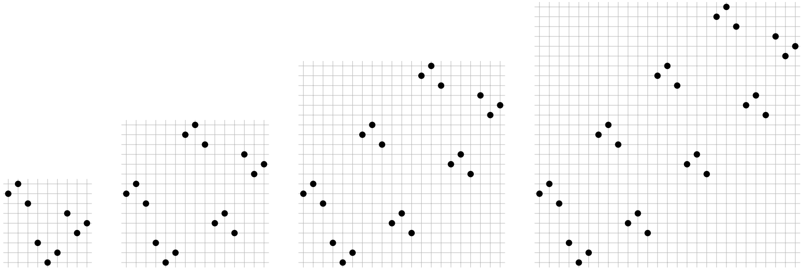

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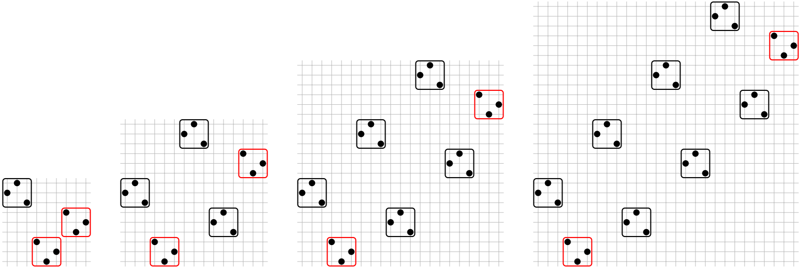

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The proof needs several ingredients:

- The substitution decomposition (a.k.a. modular decomposition)
- Nash-Williams' 1963 minimal bad sequence argument
- Gallai's 1967 characterization of the 'preimages' of G_π,

$$\Sigma_\pi = \{\text{permutations } \sigma : G_\sigma \cong G_\pi\}.$$

  We restrict to simple permutations, in which case $|\Sigma_\pi| \leq 4$.
- A 2019 result of Klavík and Zeman concerning automorphism groups of prime inversion graphs.

Some open questions and conjectures

Conjecture

If the permutation class $\mathcal{C}^{+1}$ is WQO, then $\mathcal{C}$ (and thus also $\mathcal{C}^{+1}$) is LWQO.

n-WQO: WQO when using a set of n incomparable labels.

Conjecture (Pouzet 1972)

A class of graphs is 2-WQO if and only if it is n-WQO for every $n \geq 2$.

Question

Is every 2-WQO permutation class also LWQO?

Thanks!