Well-quasi-ordering permutations

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Joint work with Vince Vatter (U. Florida)

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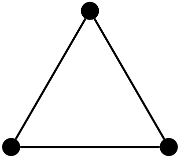
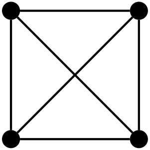
The Open University

Embedding graphs

Can I delete vertices (and incident edges) from the graph on the right to produce the graph on the left?





Embedding graphs

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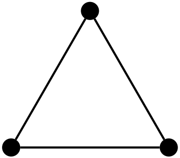
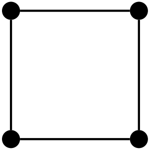


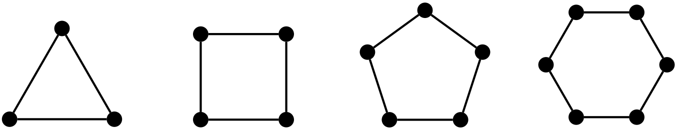


Embedding graphs

Can I delete vertices (and incident edges) from the graph on the right to produce the graph on the left?

No!

Is any graph in the following (infinite) list an induced subgraph of another?



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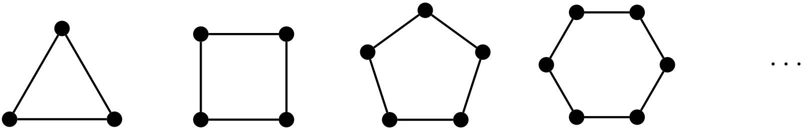

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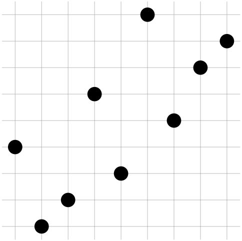

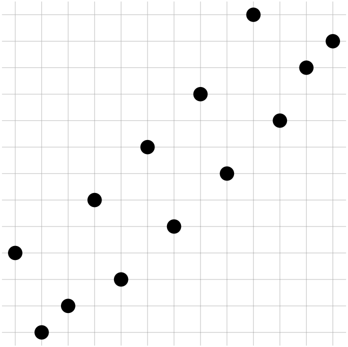

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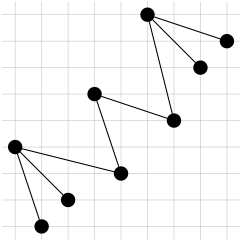

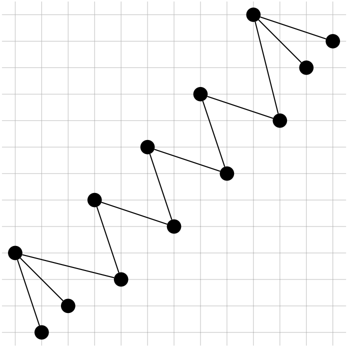

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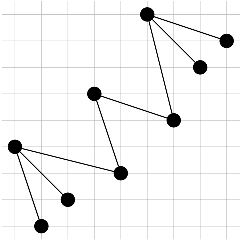

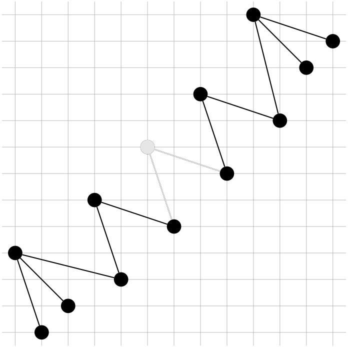

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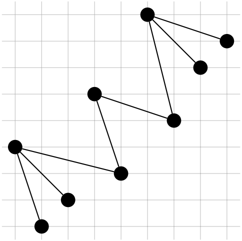

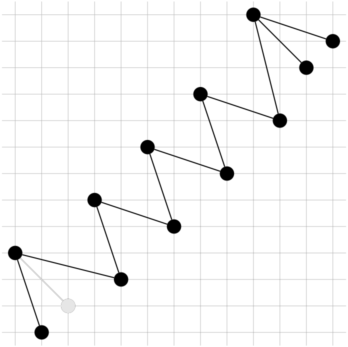

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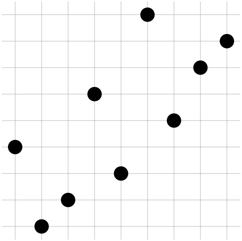
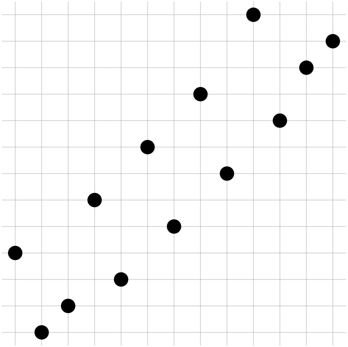
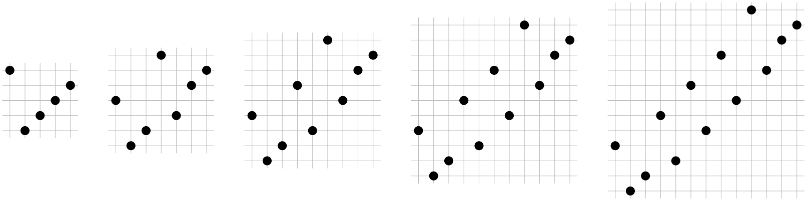




No!

No permutation in the following list embeds in any other



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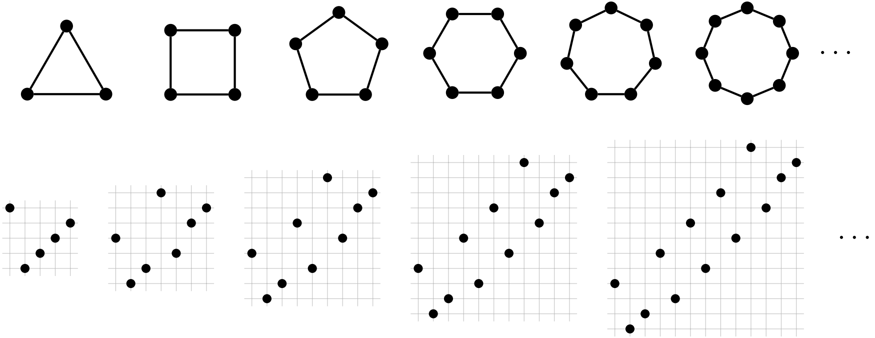
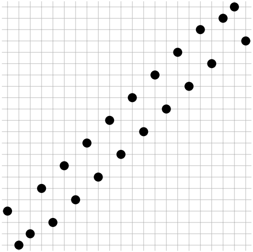
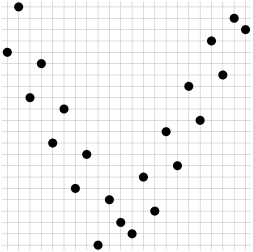
Infinite antichain: An infinite set of combinatorial structures such that no one embeds in another.



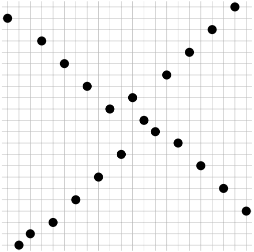
A library of infinite antichains of permutations



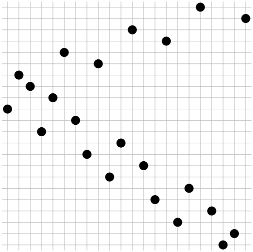
$\mathfrak{Osc}$

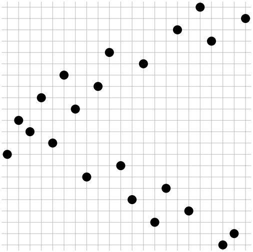


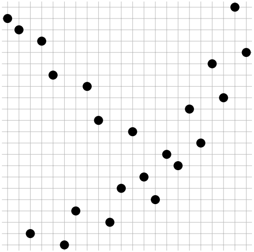
$\mathfrak{V}$



$\mathfrak{W}$







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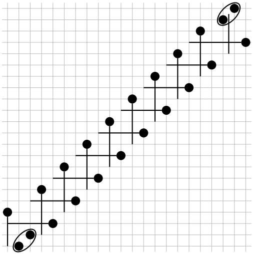

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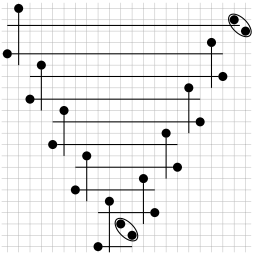
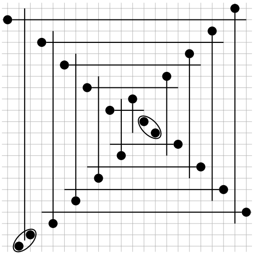


$\mathfrak{V}$



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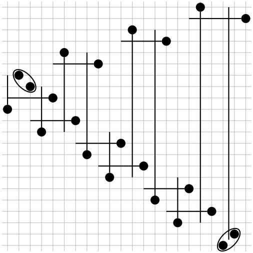
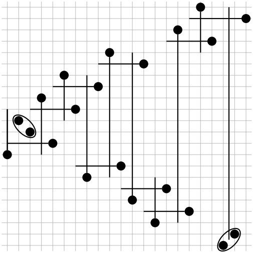
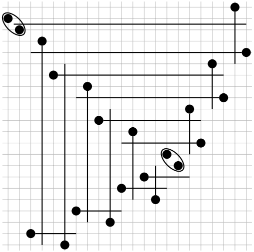
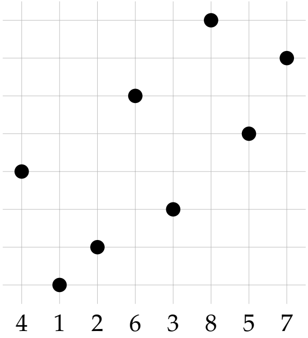


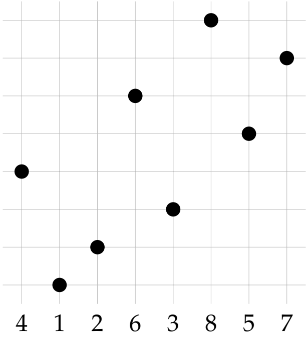




§1 Permutation containment



- Think of the n entries of $\pi = \pi(1) \cdots \pi(n)$ as *vertices*



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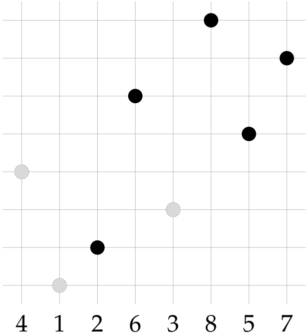
- Containment ordering: ‘Delete entries, and rescale’



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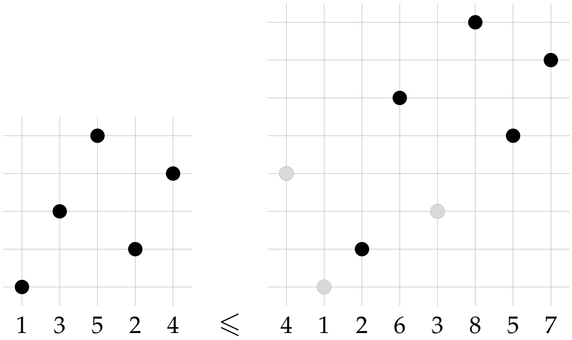

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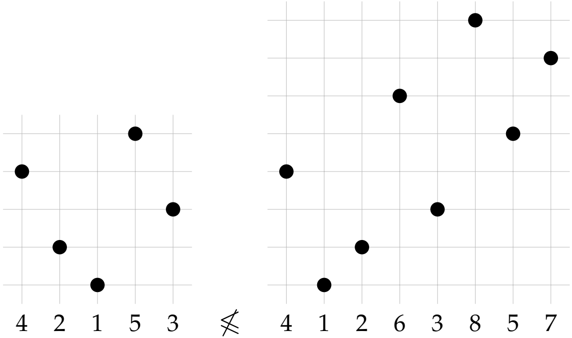
- Formally: $\sigma \leqslant \pi$ if π has a subsequence with the same relative ordering as σ.



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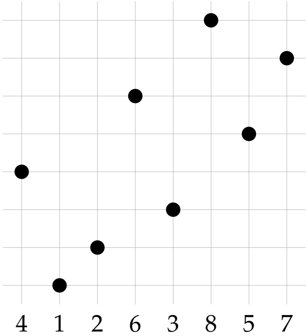
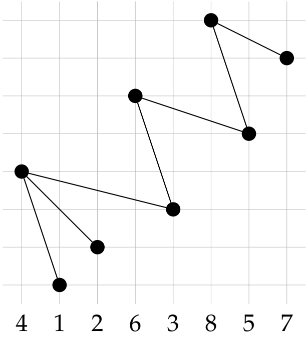
- If $\sigma \nleqslant \pi$, then π avoids σ.



Inversion graph G_π of $\pi = \pi(1) \cdots \pi(n)$:

- Vertices $= \{1, 2, \ldots, n\}$
- Edges: $a \sim b$ if $a < b$ and $\pi(b) < \pi(a)$ (edges $=$ inversions)



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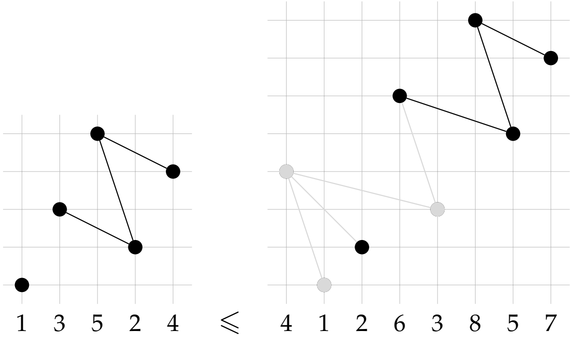

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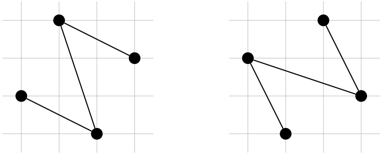
Induced substructure preserved: $\sigma \leqslant \pi$ implies $G_\sigma \leqslant_{\text{ind}} G_\pi$

Permutations to graphs is many-to-one

$\sigma \leqslant \pi$ implies $G_\sigma \leqslant_{\text{ind}} G_\pi$ but:



$G_{2413} \cong G_{3142} \cong$  even though $2413 \neq 3142$.

Hereditary classes

Set $\mathcal{C}$ of graphs/permutations is hereditary if
$A \in \mathcal{C}$ and B is an induced substructure of A, then $B \in \mathcal{C}$. ('class')

Every hereditary class has a unique set of minimal forbidden elements:
the smallest things that are 'not in the class'. ('basis')

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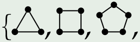
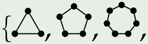
Some graph classes

Class $\mathcal{C} = \text{Free}(\mathfrak{B})$	Basis $\mathfrak{B}$
Empty graphs (no edges)	{  }
Forests	{ , ... }
Bipartite graphs	{ , ... }
Inversion graphs	$\text{Free}(C_{n+4}, T_2, X_2, X_3, X_{30}, X_{31}, X_{32}, X_{33}, X_{34}, X_{36}, XF_1^{2n+3}, XF_2^{n+1}, XF_3^{n}, XF_4^{n}, XF_5^{2n+3}, XF_6^{2n+2}, +\text{complements})$ (Gallai 1967)

Hereditary classes

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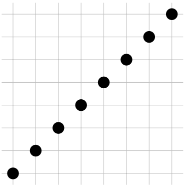
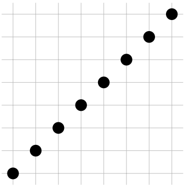
Some permutation classes

Class $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$	Basis $\mathfrak{B}$
$\{1, 12, 123, \ldots\}$	$\{21\}$
Union of 2 increases	$\{321\}$
'Stack sortable'	$\{231\}$
'2-stack-sortable'	Infinite (Murphy 2003)

§2 Counting classes



$\text{Av}(21) = \{1, 12, 123, \ldots\}$ has 1 permutation of each length.



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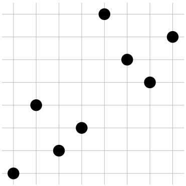
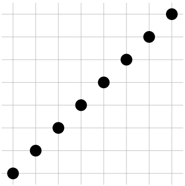


$\mathrm{Av}(231)$ has $1, 2, 5, 14, 42, \ldots$ of lengths $n = 1, 2, 3, 4, 5, \ldots$.



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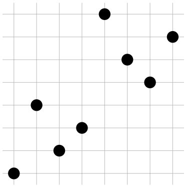

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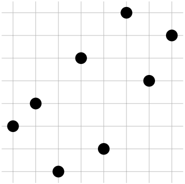
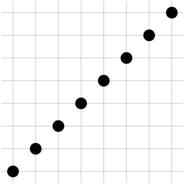
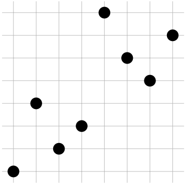


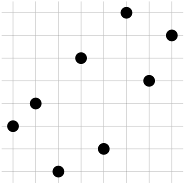
$\mathrm{Av}(321)$ has $1, 2, 5, 14, 42, \ldots$ of lengths $n = 1, 2, 3, 4, 5, \ldots$.



$\mathrm{Av}(21) = \{1, 12, 123, \ldots\}$ has 1 permutation of each length.



$\mathrm{Av}(231)$ has $1, 2, 5, 14, 42, \ldots$ of lengths $n = 1, 2, 3, 4, 5, \ldots$.



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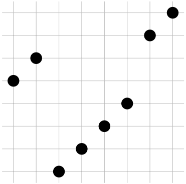


$\mathrm{Av}(132, 321)$ has $\binom{n}{2} + 1$ of length n.

Typical questions in Permutation Patterns

For a permutation class $\mathcal{C}$:

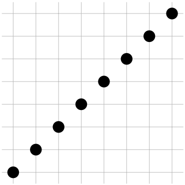
- What is the generating function? (e.g. rational, algebraic, *D*-finite)

$$f_{\mathcal{C}}(z) = \sum_{\pi \in \mathcal{C}} z^{|\pi|} = \sum_{n=1}^{\infty} |\mathcal{C}_n| z^n, \quad \text{where } \mathcal{C}_n = \{\pi \in \mathcal{C} : |\pi| = n\}.$$

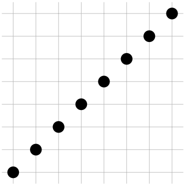
- What is the growth rate?

$$\mathrm{gr}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}, \quad \text{which exists by Marcus \& Tardos (2004).}$$

- What is the basis? (Is $\mathcal{C}$ finitely based?)
- What do the permutations 'look like'?



$$f_{\mathrm{Av}(21)} = \frac{1}{1-z}; \quad \mathrm{gr} = 1.$$



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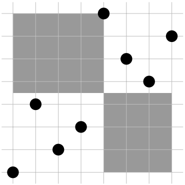
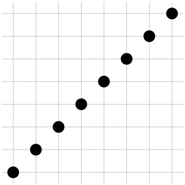


$$f_{\mathrm{Av}(231)} = \frac{1-\sqrt{1-4z}}{2z}; \quad \mathrm{gr} = 4.$$



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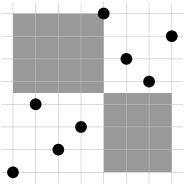

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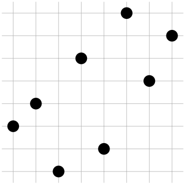
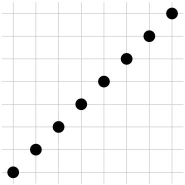
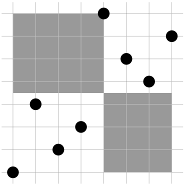


$$f_{\mathrm{Av}(321)} = \frac{1-\sqrt{1-4z}}{2z}; \quad \mathrm{gr} = 4.$$



$$f_{\mathrm{Av}(21)} = \frac{1}{1-z}; \quad \mathrm{gr} = 1.$$



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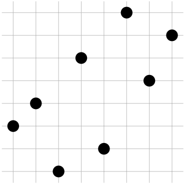

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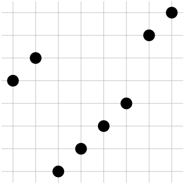


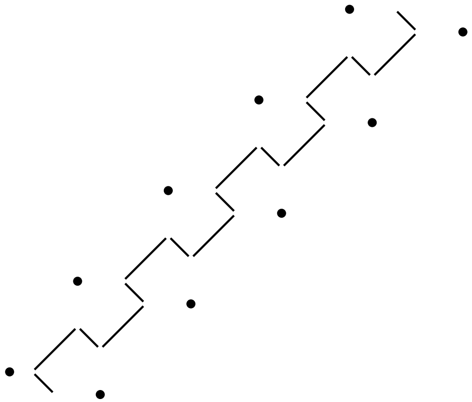
$$f_{\mathrm{Av}(132,123)} = \frac{1-z+z^2}{(1-z)^3}; \quad \mathrm{gr} = 1.$$

‘Tame’ structure tends to give a ‘tame’ generating function.

Theorem (Albert, B., 2014)

The permutation class that determine Schubert varieties defined by inclusions ($Av(4231, 31524, 42513, 351624)$) has generating function

$$\frac{1 - 3z - 2z^2 - (1 - z - 2z^2)\sqrt{1 - 4z}}{1 - 3z - (1 - z + 2z^2)\sqrt{1 - 4z}}.$$



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but not *vice versa*…

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Every proper permutation class $\mathcal{C}$ is contained in a permutation class with a rational generating function.

'Proof'.

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such that $\mathrm{Av}(\mathfrak{A}) \cup \mathfrak{A}$ has a rational generating function.
Union this with $\mathcal{C}$, and remove enough antichain elements of each length to preserve rationality. □

‘Tame’ structure tends to give a ‘tame’ generating function.

It is tempting to generalise…

Conjecture (Noonan, Zeilberger, 1996)

Every finitely based class has a D-finite generating function.

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So 'finitely based' isn't universally tame. Nevertheless…

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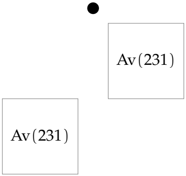
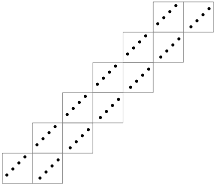
Conjecture

Every finitely based class with growth rate < 4 has a rational generating function.

‘Tame’ structure tends to give a ‘tame’ generating function.

Is there something general we can say here?

§3 Well-quasi-ordering

	Av(231)	Av(321)
Growth rate	4	4
Generating function	$\frac{1-\sqrt{1-4z}}{2z}$	$\frac{1-\sqrt{1-4z}}{2z}$
Basis	231	321
‘Look like’		

What about *subclasses* of Av(231), Av(321)?

	$\mathcal{C} \subsetneq \text{Av}(231)$	$\mathcal{D} \subsetneq \text{Av}(321)$
Growth rate	Countably many possibilities	Includes $[2.36, 2.48]$ (Bevan, 2018)
Generating function	Rational (Albert, Atkinson, 2005)	Could be anything
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Generating function	Rational (Albert, Atkinson, 2005)	Could be anything
Basis	Finite	Finite or infinite
Infinite antichains?	No	Yes:  …

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A strong indicator of ‘tameness’, for example, even though Av(321) is not WQO:

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Every WQO or finitely based subclass of Av(321) *has a rational generating function.*

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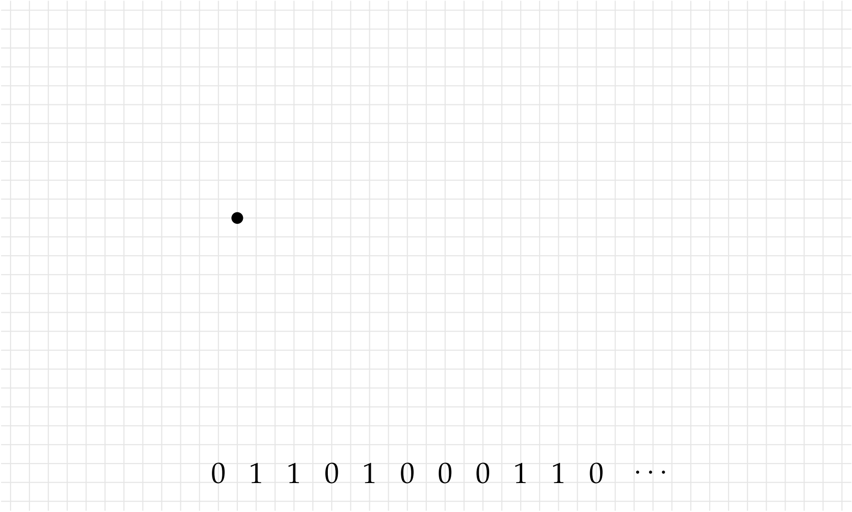
This also turned out to be a generalisation too far…

0 1 1 0 1 0 0 0 1 1 0 ⋯

Infinite binary sequence $s \longrightarrow$ infinite permutation $\pi_s : \mathbb{N} \to \mathbb{Z}$



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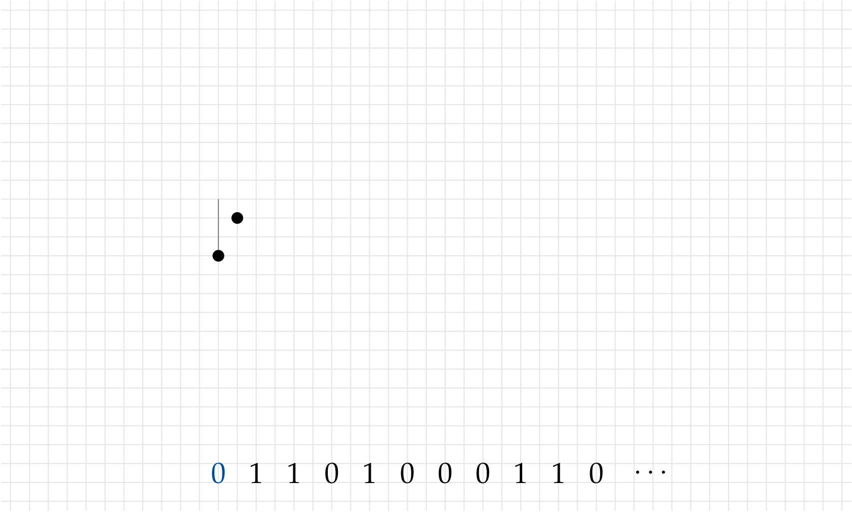

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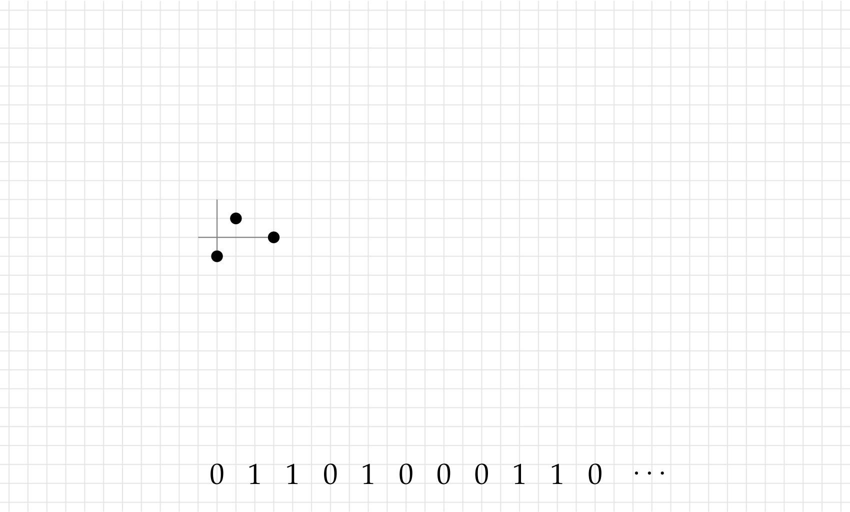

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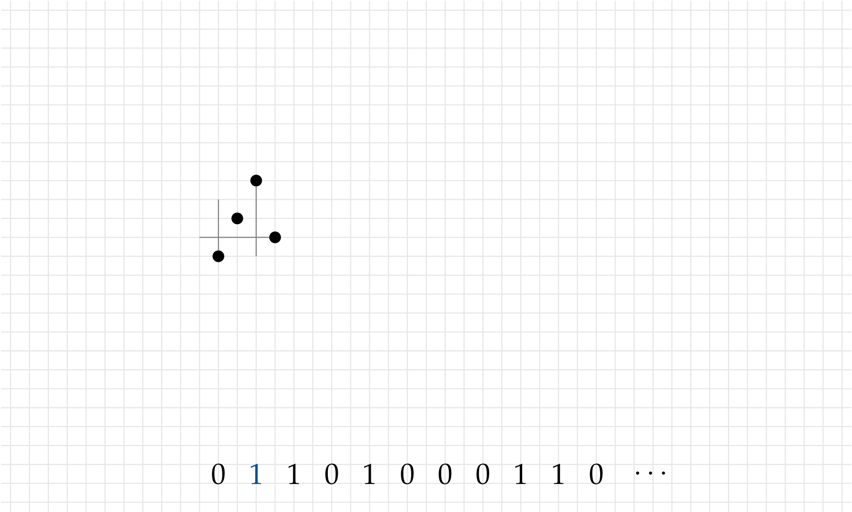

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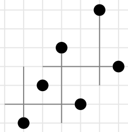

0 1 1 0 1 0 0 0 1 1 0 $\cdots$

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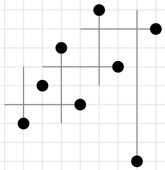

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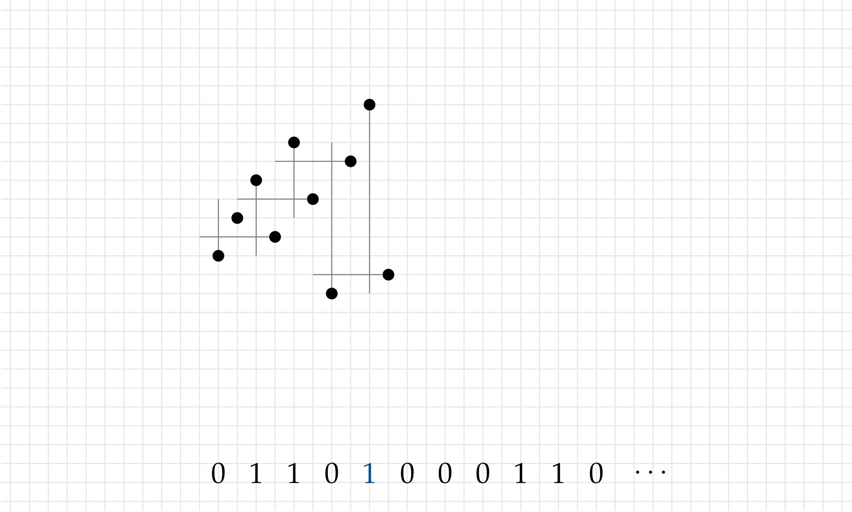

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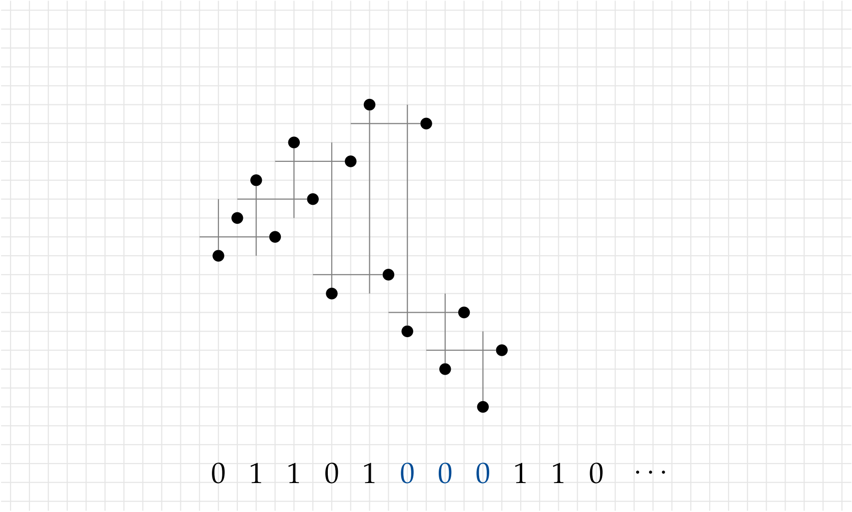

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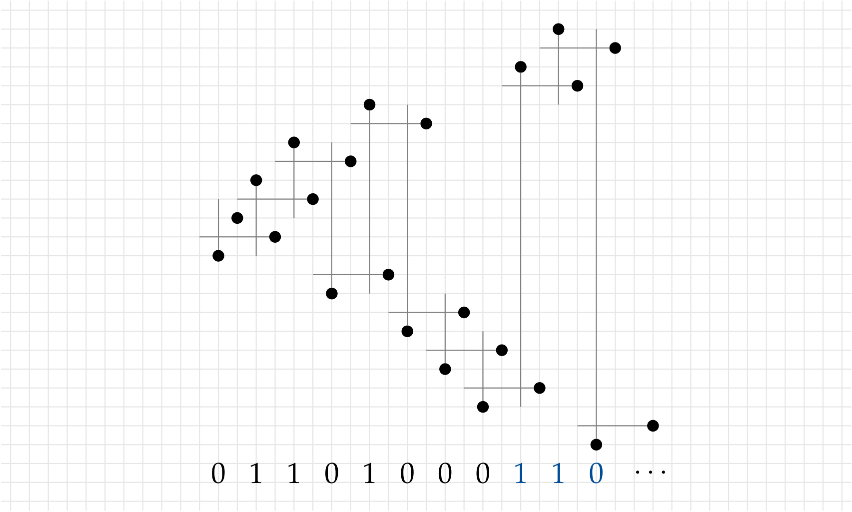

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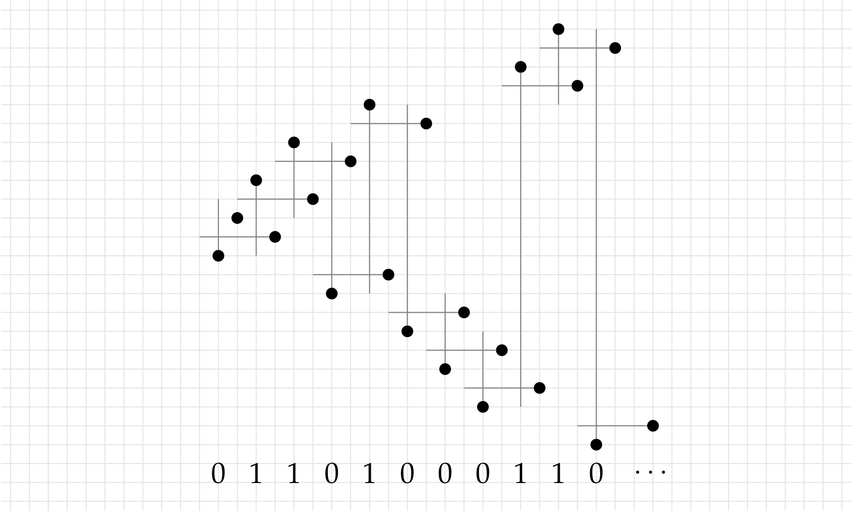

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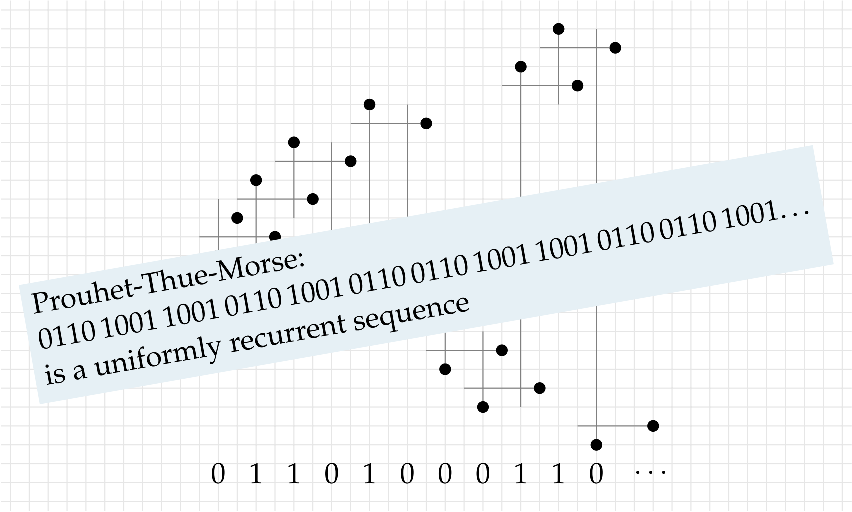
class $\mathcal{C}_s = \{$(finite) permutations $\pi \leqslant \pi_s\}$



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Sequence construction (generalising Prouhet-Thue-Morse) from Maurice Pouzet's 1978 thesis $\Longrightarrow$ uncountably many WQO classes with *different* generating functions.

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Theorem (B., Vatter, 2023+)

There are uncountably many distinct enumerations of WQO permutation classes.

Hence, not all WQO classes have algebraic (or D-finite) generating functions.

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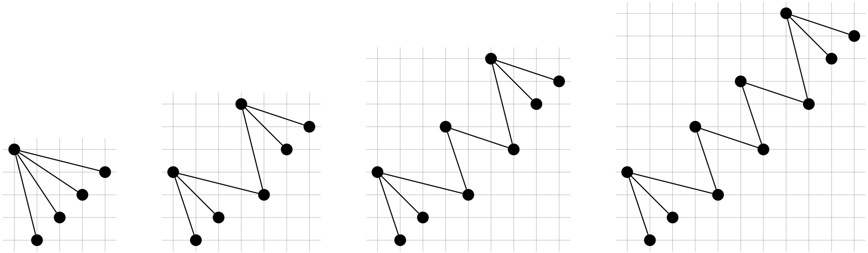
§4 Labelled WQO

A regular infinite antichain:



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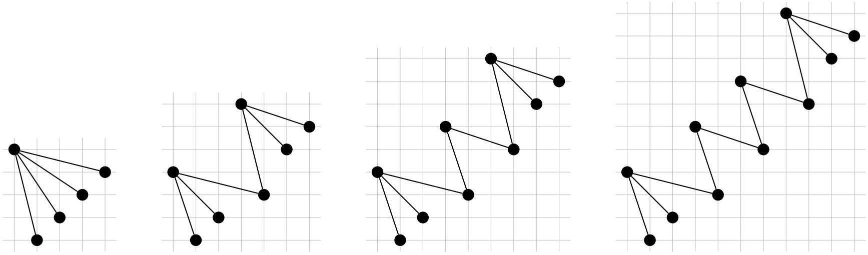
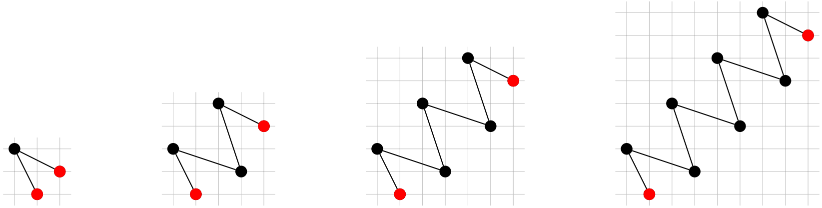


A labelled infinite antichain:



Labels can be (partially) ordered (e.g. • $\preceq$ •): embedding must respect the label ordering.

Labelled WQO

A class is labelled well-quasi-ordered (LWQO) if we cannot construct a labelled infinite antichain, no matter the set of labels.†

† Includes infinite sets of labels, but they must be WQO.

Theorem (After Pouzet, 1972)

LWQO (permutation) classes must be finitely based.

Corollary

There are only countably many LWQO permutation classes.

‘Tame’ structure tends to give a ‘tame’ generating function.

Does LWQO guarantee tame enumeration?

§5 Permutations & inversion graphs

Does WQO translate?

Recall: $\sigma \leqslant \pi \Rightarrow G_\sigma \leqslant_{\text{ind}} G_\pi$.

Thus $\mathcal{C}$ (L)WQO $\Rightarrow G_{\mathcal{C}}$ (L)WQO.

Question

If $\mathcal{C}$ is a permutation class such that $G_{\mathcal{C}}$ is WQO, must $\mathcal{C}$ be WQO?

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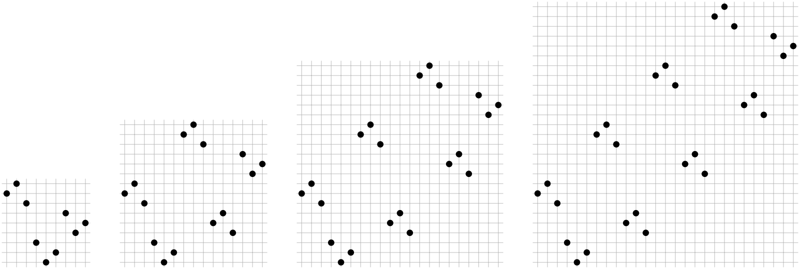
This question seems to be very difficult. Here is a permutation antichain which turns into a chain of graphs:



Note that $G_{231} \cong G_{312} \cong$ 

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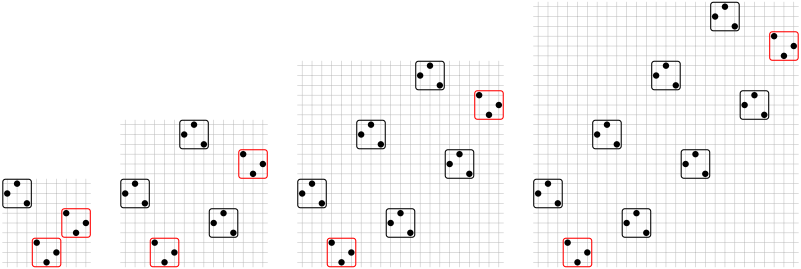

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Theorem (B., Vatter, 2022)

Let $\mathcal{C}$ be a permutation class. $\mathcal{C}$ is LWQO if and only if $G_{\mathcal{C}}$ is LWQO.

Thanks!