Antichains and the Structure of Permutation Classes

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Outline

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1. Introduction
   - Permutation classes
   - Enumeration
   - Partial well-order and antichains
2. Simple permutations
   - Intervals
   - Substitution decomposition
   - Finitely many simples
3. Grid classes
   - Introduction
   - Monotone classes and partial well-order
   - Far beyond monotone
   - Nearly monotone
4. Summary

Setting the Scene

- Permutation of length n: an ordering on the symbols $1, \ldots, n$.
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- Graphical viewpoint: plot the points $(i, \pi(i))$.

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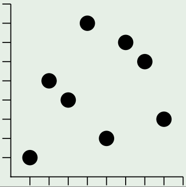

Stack Sorting

- Knuth (1969): what permutations can be sorted through a stack?

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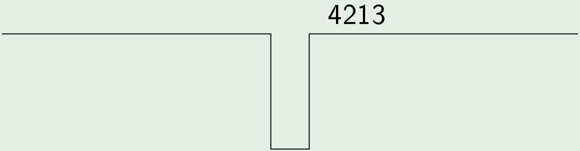

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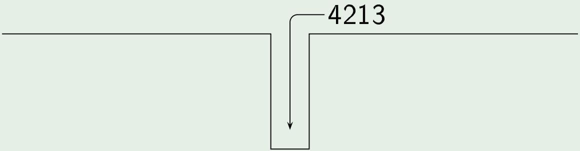

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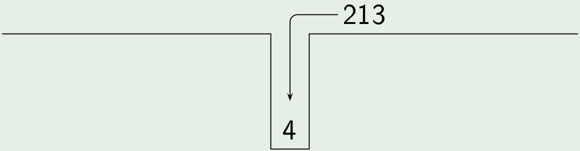

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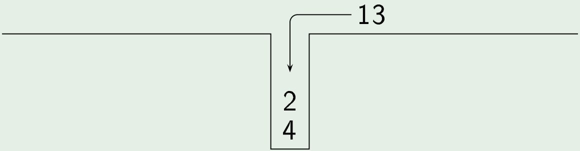

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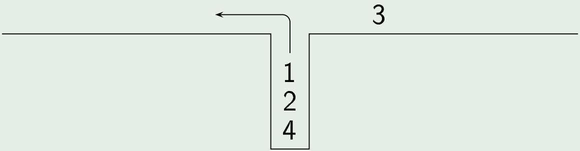

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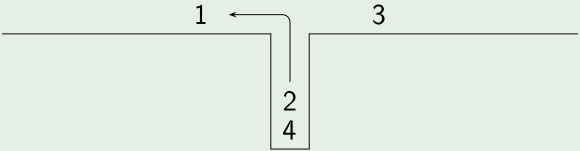

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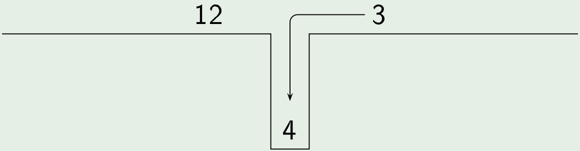

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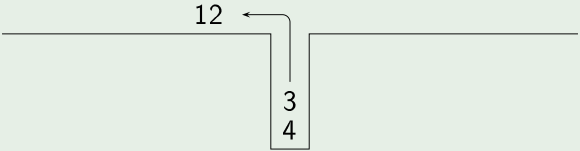

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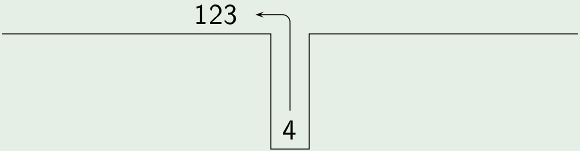

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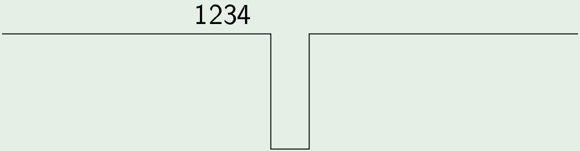

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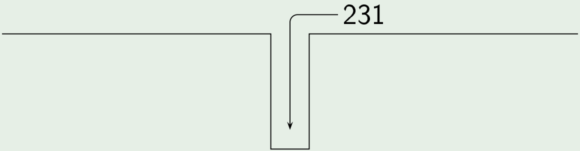

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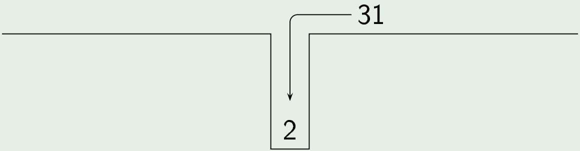

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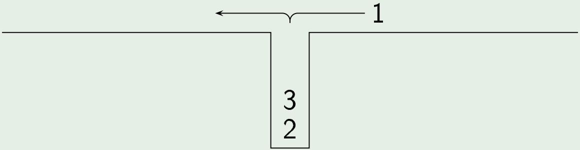

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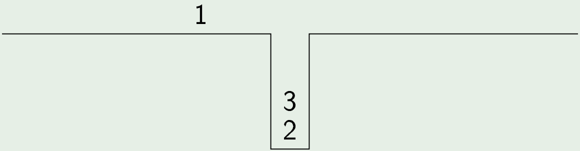

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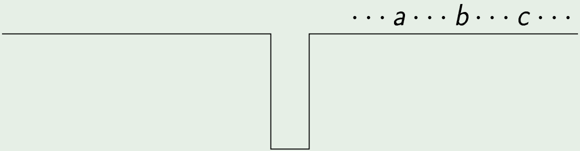

- 231 is not stack-sortable.
- In general: can't sort any permutation with a subsequence *abc* such that $c < a < b$. (*abc* forms a 231 "pattern".)

Containment

- A permutation $\tau = \tau(1) \cdots \tau(k)$ is contained in the permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ if there exists a subsequence $\sigma(i_1)\sigma(i_2) \cdots \sigma(i_k)$ order isomorphic to τ.

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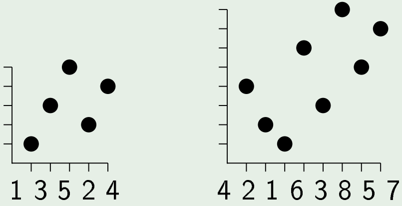

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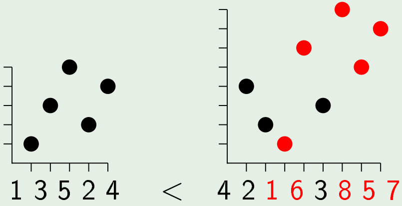

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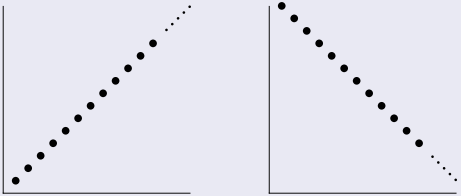
- E.g. the stack-sortable permutations are $\mathrm{Av}(231)$.
- Graph theoretic analogue: hereditary properties of graphs (e.g. triangle-free graphs, planar graphs, ...).

Easy Examples

- $\mathrm{Av}(21) = \{1, 12, 123, 1234, \ldots\}$, the increasing permutations.
- $\mathrm{Av}(12) = \{1, 21, 321, 4321, \ldots\}$, the decreasing permutations.

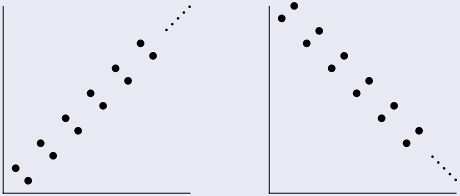
Typical Elements



Easy Examples

- $\oplus 21 = \mathrm{Av}(321, 312, 231) = \{1, 12, 21, 123, 132, 213, \ldots\}$.
- $\ominus 12 = \mathrm{Av}(123, 213, 132) = \{1, 12, 21, 231, 312, 321, \ldots\}$.

Typical Elements



Exact Enumeration

- $\mathcal{C}_n$ – permutations in $\mathcal{C}$ of length n.
- $\sum |\mathcal{C}_n| x^n$ is the generating function.

Example

The generating function of $\mathcal{C} = \mathrm{Av}(12)$ is:

$$1 + x + x^2 + x^3 + \cdots \quad = \quad \frac{1}{1-x}$$

Asymptotic Enumeration

Theorem (Marcus and Tardos, 2004)

For every permutation class $\mathcal{C}$ other than the class of all permutations, there exists a constant K such that

$$|\mathcal{C}_n| \leq K^n$$

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- Upper growth rate of $\mathcal{C}$ is $\limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$.
- Big open question: does the growth rate, $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$, always exist?

Av(321) vs Av(231)

- Stack sortable permutations $\mathrm{Av}(231)$ enumerated by the Catalan numbers. Generating function:

$$f(x) = \frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

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- Using the Robinson-Schensted-Knuth correspondence with Young Tableaux, $|\mathrm{Av}(321)|_n = |\mathrm{Av}(231)|_n$.
- Despite being equinumerous, these two classes are very different: $\mathrm{Av}(321)$ contains infinite antichains and hence has uncountably many subclasses, while $\mathrm{Av}(231)$ does not.

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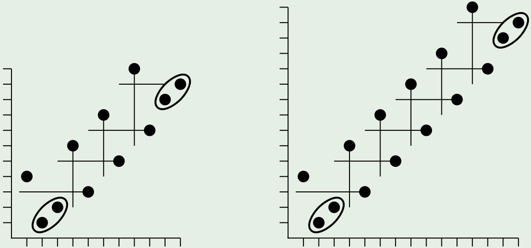

- N.B. These permutations avoid 321.

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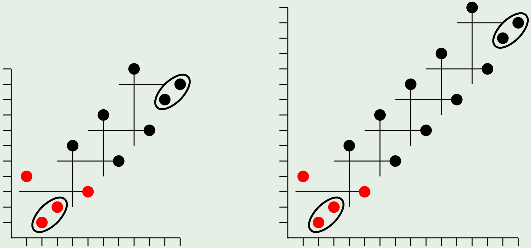

- Bottom copies of 4123 must match up: the anchor.

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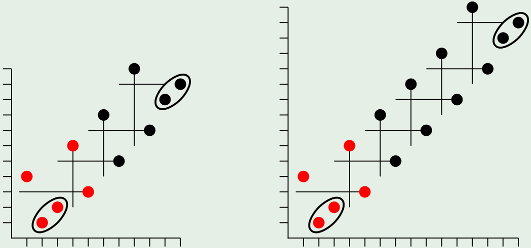

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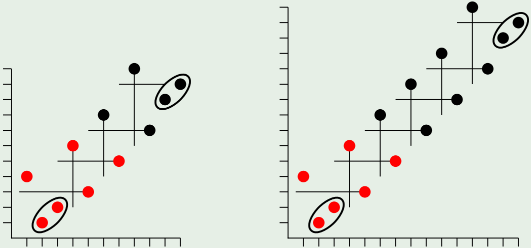

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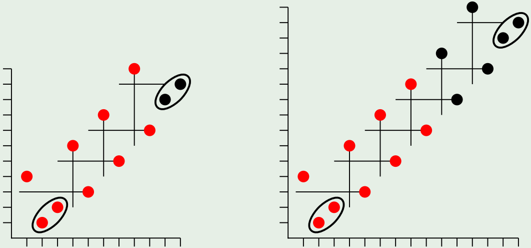

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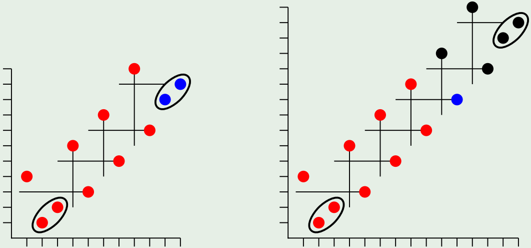

- Last pair cannot be embedded.

When are there antichains?

No infinite antichains.

- Words over a finite alphabet [Higman].
- Graphs closed under minors [Robertson and Seymour].

Infinite antichains.

- Graphs closed under induced subgraphs (or merely subgraphs). e.g. $C_3, C_4, C_5, \ldots$
- Permutations closed under containment.
- Tournaments, digraphs, ...

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Can we decide whether a permutation class given by a finite basis is pwo?

- To prove pwo — Higman's theorem is useful.
- To prove not pwo — find an antichain.

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Question

Can we decide whether a hereditary property given by a finite basis is wqo?

- To prove pwo — Higman's theorem is useful.
- To prove not pwo — find an antichain.
- Other structures: well quasi-order, not pwo, but same idea.

Outline

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- Pick any permutation π.
- An interval of π is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

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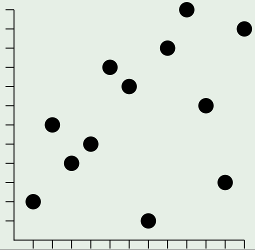

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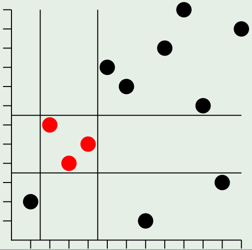

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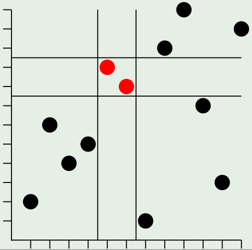

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- Intervals are important in biomathematics (genetic algorithms, matching gene sequences).

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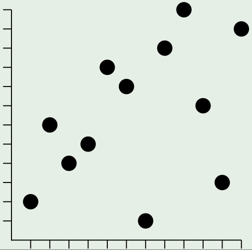

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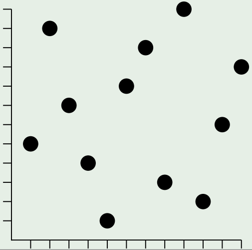

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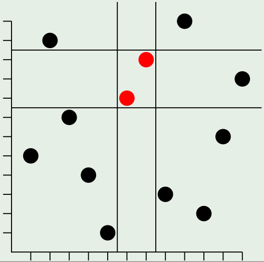

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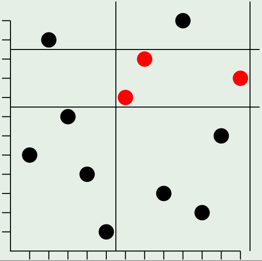

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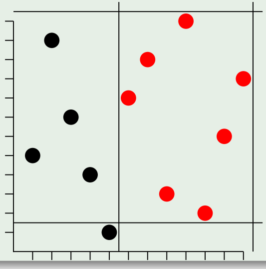

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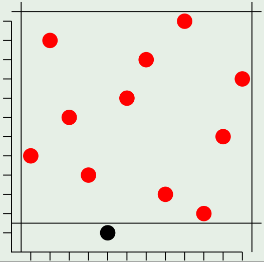

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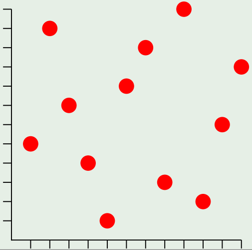

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- Two of length four: 2413 and 3142.
- The sequence goes $1, 2, 0, 2, 6, 46, 338, 2926, 28146, \ldots$

Theorem (Albert, Atkinson and Klazar, 2003)

The number of simple permutations is asymptotically given by

$$\frac{n!}{e^2}\left(1 - \frac{4}{n} + \frac{2}{n(n-1)} + O(n^{-3})\right).$$

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- Frequently rediscovered in different settings under various names: modular decomposition, disjunctive decomposition, X-join...
- Möhring (1985), and Möhring and Radermacher (1984) discuss applications in combinatorial optimisation and game theory.

Decomposing Permutations

- Break permutation into maximal proper intervals.

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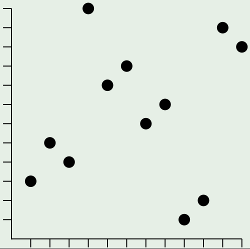

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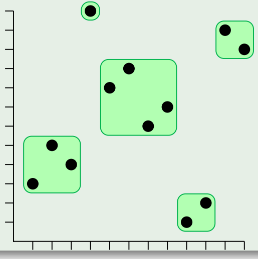

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- Break permutation into maximal proper intervals.
- Gives a unique simple permutation, the skeleton.

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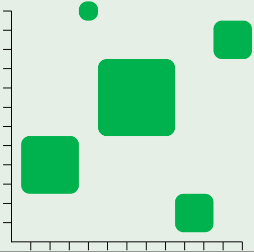

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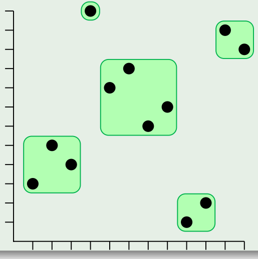

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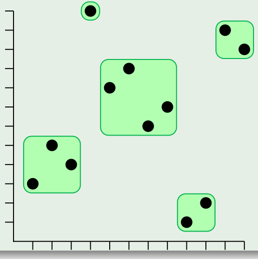
- This decomposition is the substitution decomposition.

Example



Non-uniqueness

- Simple permutation of length 2: block decomposition is not unique.

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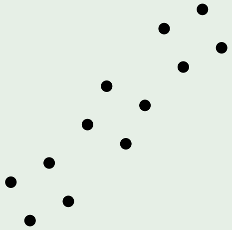

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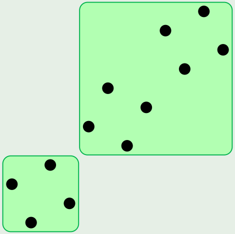

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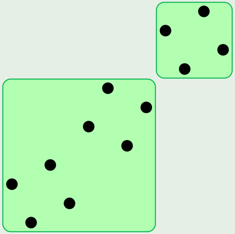
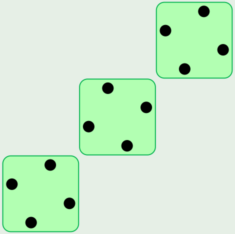


Non-uniqueness

- Underlying structure is an increasing permutation.

Example



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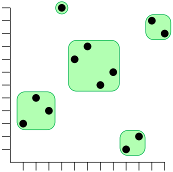
Theorem (B., Ruškuc and Vatter, 2008)

It is possible to decide whether a permutation class given by a finite basis contains infinitely many simple permutations.

Finitely Many Simples ⇒ Partially Well-Ordered



- Take a class $\mathcal{C}$ containing a finite set S of simple permutations.
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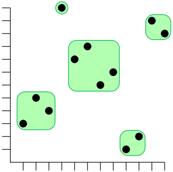

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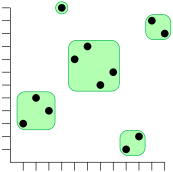

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- Now use Higman's Theorem.

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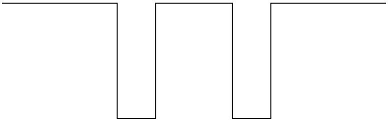
- $\mathrm{Av}(231)$ contains only the simples 12 and 21, and so it is partially well-ordered.
- Little is known about two-stack-sortable permutations: they are not finitely based.

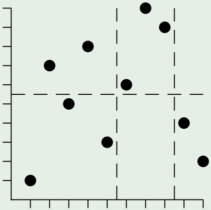


Outline

- Matrix $\mathcal{M}$ whose entries are permutation classes.
- $\mathrm{Grid}(\mathcal{M})$ the grid class of $\mathcal{M}$: all permutations which can be "gridded" so each cell satisfies constraints of $\mathcal{M}$.

Example

- Let $\mathcal{M} = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(231) & \emptyset \\ \mathrm{Av}(123) & \emptyset & \mathrm{Av}(12) \end{pmatrix}$.



$\in \mathrm{Grid}(\mathcal{M})$

Grid classes are useful

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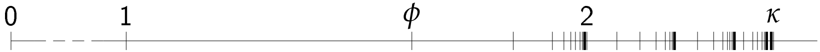

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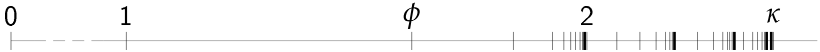


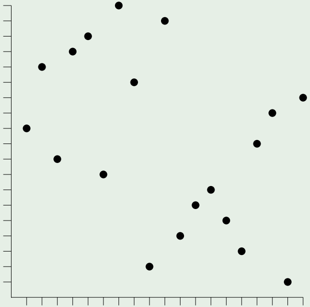
- κ is the lowest growth rate where we encounter infinite antichains, and hence uncountably many permutation classes.
- Cf "canonical properties" of graphs [Balogh, Bollobás and Weinreich].

Monotone Grid Classes

- Special case: all cells of $\mathcal{M}$ are $\mathrm{Av}(21)$ or $\mathrm{Av}(12)$.
- Rewrite $\mathcal{M}$ as a matrix with entries in $\{0, 1, -1\}$.

Example

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



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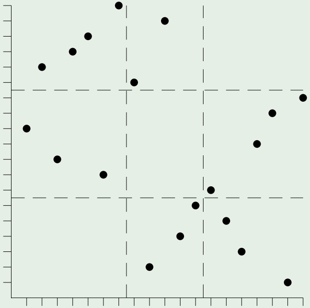

The Graph of a Matrix

- Graph of a matrix, $G(\mathcal{M})$, formed by connecting together all non-zero entries that share a row or column and are not "separated" by any other nonzero entry.

Example

$$\begin{pmatrix} \mathcal{C} & 0 & 0 & \mathcal{D} \\ 0 & 0 & \mathcal{E} & 0 \\ \mathcal{D} & \mathcal{E} & 0 & \mathcal{C} \\ 0 & 0 & 0 & \mathcal{D} \end{pmatrix}$$

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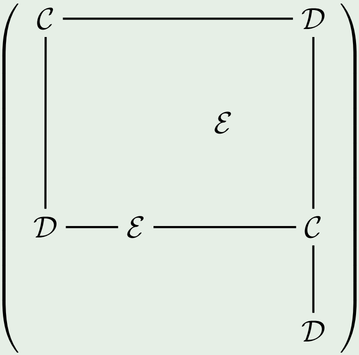

Monotone Grids and Partial Well-Order

Theorem (Murphy and Vatter, 2003)

The monotone grid class $\mathrm{Grid}(\mathcal{M})$ *is pwo if and only if* $G(\mathcal{M})$ *is a forest, i.e.* $G(\mathcal{M})$ *contains no cycles.*

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Proof.

($\Leftarrow$) New shorter proof in Waton's Thesis (2007).

1	-1	
	1	
-1		-1

$\square$

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$(\Leftarrow)$ Partial multiplication table.

-1	1	-1	
1		1	
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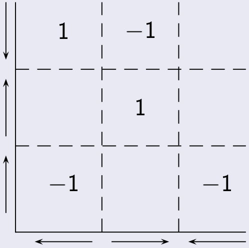
$(\Leftarrow)$ ± 1 correspond to directions.



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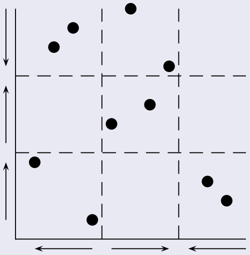
Proof.

($\Leftarrow$) Form one order per arrow.



- $1 < 9 < 8 < 4$.
- $5 < 10 < 6 < 7$.
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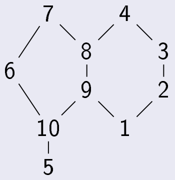
Proof.

($\Leftarrow$) No cycles, so this gives a poset.



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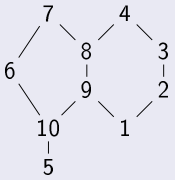
$(\Leftarrow)$ Linear extension: $5 < 10 < 1 < 9 < 2 < 6 < 8 < 3 < 7 < 4$



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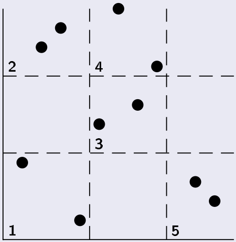

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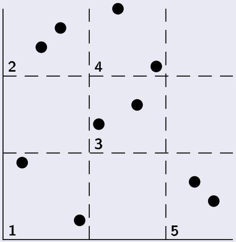

- Encode by region: 3412532541.
- Higman's Theorem: $\{1, 2, 3, 4, 5\}^*$ is pwo under the subword order.
- Encoding is reversible, hence $\mathrm{Grid}(\mathcal{M})$ is pwo.

□

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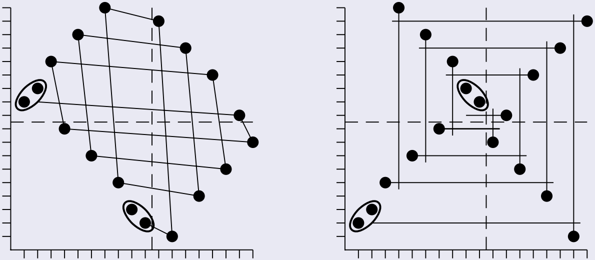
Proof.

($\Rightarrow$) Construct fundamental antichains that "walk" around a cycle.



□

When does that apply?

Question

When is a class $\mathcal{C}$ (a subset of) a monotone grid class?

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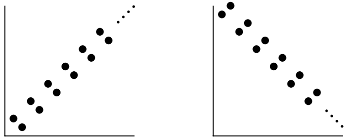
Answer [Vatter]

A class $\mathcal{C}$ is monotone griddable if and only if it contains neither the classes $\oplus 21$ nor $\ominus 12$.



Non-monotone cells

- If a class is not monotone griddable, then perhaps it can be gridded by a matrix which is mostly monotone:

Example

$$\begin{pmatrix} \mathrm{Av}(21) & 0 & 0 & \mathrm{Av}(21) \\ 0 & \ominus 12 & 0 & 0 \\ \oplus 21 & 0 & \mathrm{Av}(12) & 0 \\ 0 & 0 & 0 & \oplus 21 \end{pmatrix}$$

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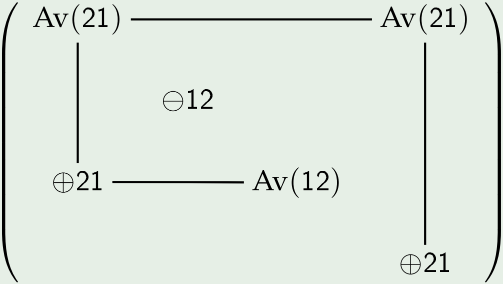


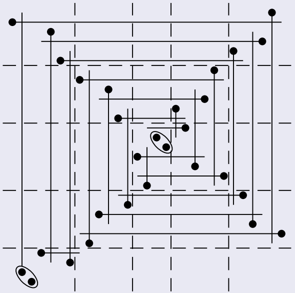
- Graph must still be a forest, but now we're interested in how many non-monotone-griddable cells lie in each component.

Two is too many

Theorem

A grid class whose graph has a component containing two or more non-monotone-griddable classes is not pwo.

Proof.



- Antichain element.

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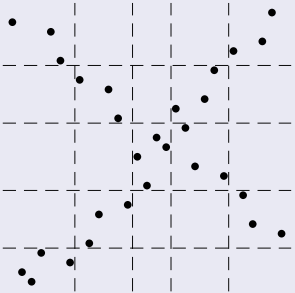

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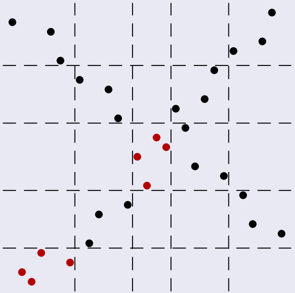

- Antichain element.
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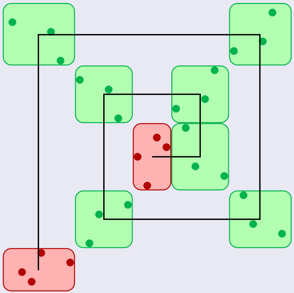

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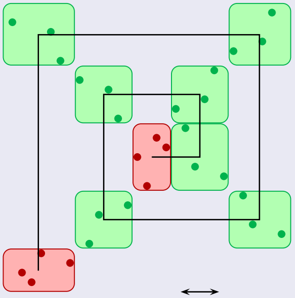

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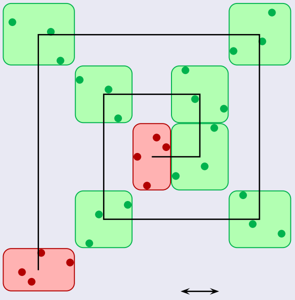

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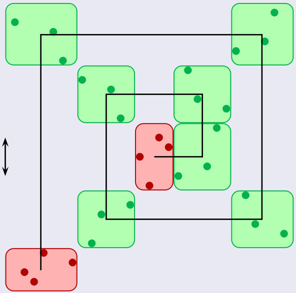

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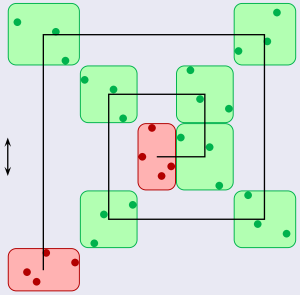

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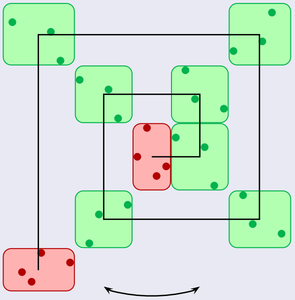

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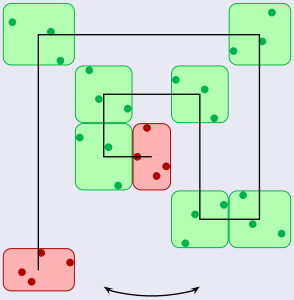

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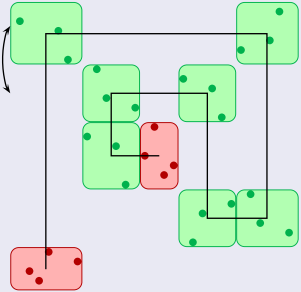

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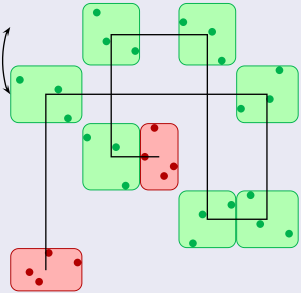

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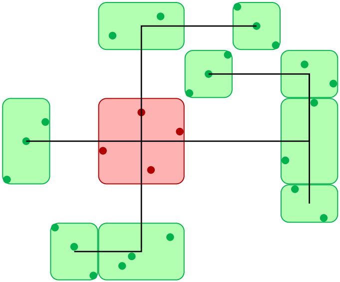
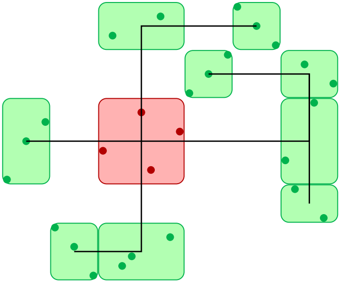
Just one non-monotone

- Suppose the bad cell contains only finitely many simple permutations.



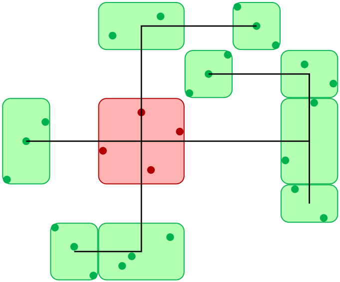
Just one non-monotone

- Suppose the bad cell contains only finitely many simple permutations.
- Build permutations component-wise: use the substitution decomposition on the red cell, and view each component as a tree rooted on this cell.



Just one non-monotone

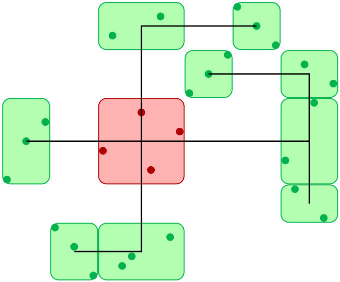
- Suppose the bad cell contains only finitely many simple permutations.
- Build permutations component-wise: use the substitution decomposition on the red cell, and view each component as a tree rooted on this cell.
- This defines a construction for all permutations in the grid class, which is amenable to Higman's Theorem.



Just one non-monotone

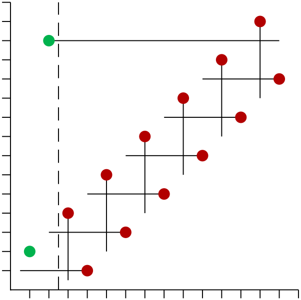
Theorem

Let $\mathcal{M}$ be a gridding matrix for which each component is a forest and contains at most one non-monotone cell. If every non-monotone cell contains only finitely many simple permutations, then $\mathrm{Grid}(\mathcal{M})$ is pwo.



But sometimes one is too much...

- One cell containing arbitrarily long increasing oscillations next to a monotone cell is bad...



Outline

Summary

- Two non-monotone per component: class not pwo.
- One non-monotone but finitely many simples: class is pwo.

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Question

Can we decide whether a permutation class given by a finite basis is pwo?

- We're closer to answering this, but still some way off.

Thanks!