Simple Extensions of Relational Structures – the Permutation Perspective

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joint work with Nik Ruškuc and Vincent Vatter

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University of St Andrews

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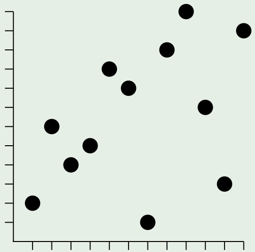
Introduction

Outline

Intervals

- Pick any permutation π.
- An interval of π is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

Example



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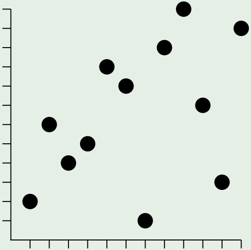

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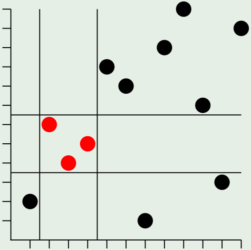

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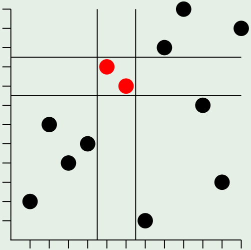

Simple Permutations

- Only intervals are singletons and the whole thing.

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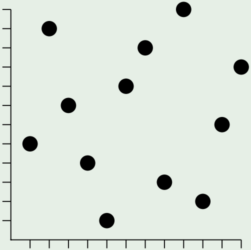

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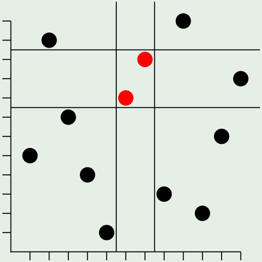

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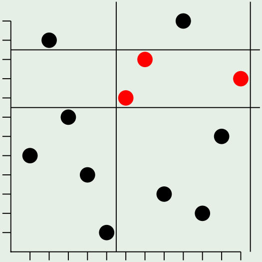

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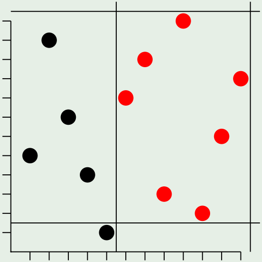

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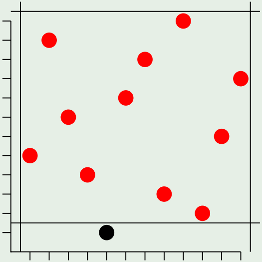

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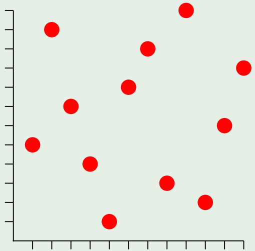

Two Binary Relations

- A relational structure: a set of points, and a set of relations on these points.

Example

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
-

Two Binary Relations

- A permutation of length n is a structure on two linear relations.

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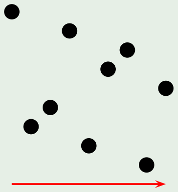

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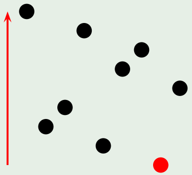

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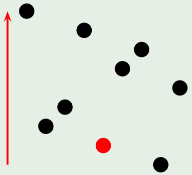

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5$

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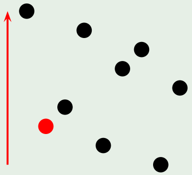

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2$

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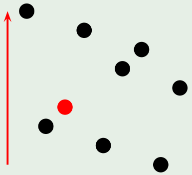

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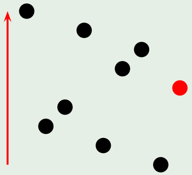

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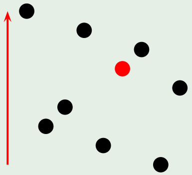

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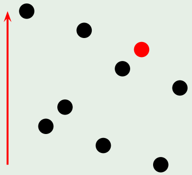

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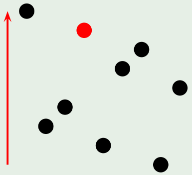

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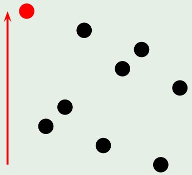

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2 \prec 3 \prec 9 \prec 6 \prec 7 \prec 4 \prec 1$

Graphs

- A graph is a relational structure on a single binary symmetric relation.
- Simple graph?

Example

- Same neighbourhood = interval.

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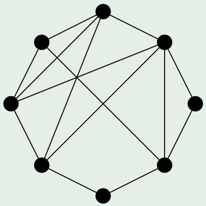

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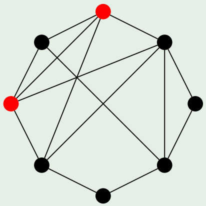

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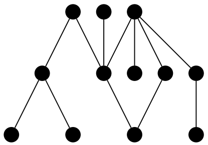
Posets

- A poset is a relational structure on a binary reflexive antisymmetric transitive relation.
- Simplicity as ever.



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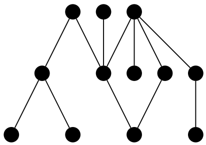
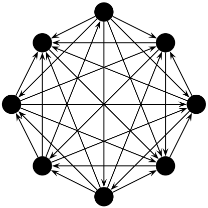


Tournaments

- A tournament is a complete oriented graph.
- As a relational structure, it is a single trichotomous binary relation. ($x \to y$, $y \to x$ or $x = y$.)
- A competition between players: $x \to y$ means "y wins."



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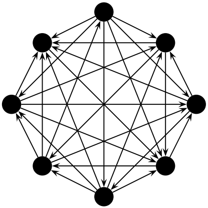

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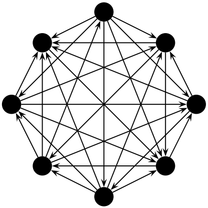

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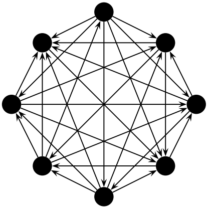

Tournaments and Algebras

- Tournament $\Longleftrightarrow$ Algebra on two idempotent binary operations.
- Simple Tournament $\Longleftrightarrow$ Simple Algebra ($=$ no non-trivial ideals).
- Can we embed an algebra into a larger simple algebra?
- How small can an embedding be?
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Two-point Simple Extensions

Theorem (Erdős, Fried, Hajnal and Milner, 1972)

Every tournament has a simple extension with at most two additional vertices.

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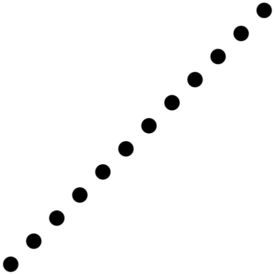
Aim

Question

How many additional points are needed to extend an arbitrary permutation to a simple one?

Two Intervals at a Time



- Worst case: an increasing permutation.

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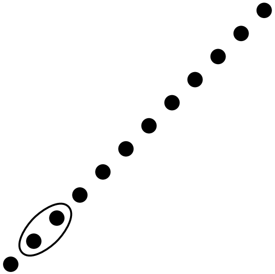
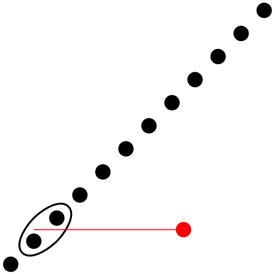


- Pick a minimal proper interval: need to “kill” it.

Two Intervals at a Time



- Kill it horizontally ...

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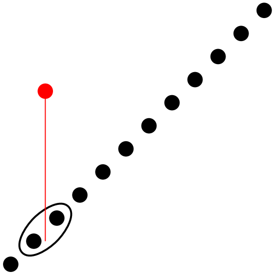

- Kill it horizontally ... or vertically.

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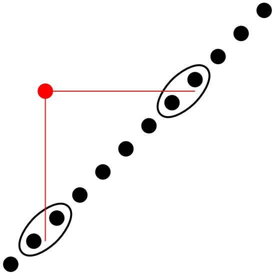

- An additional point can be used to kill two intervals.

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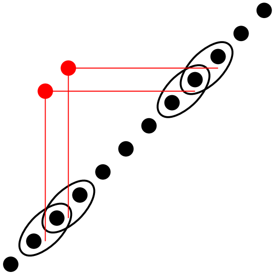

- No intervals between additional points.

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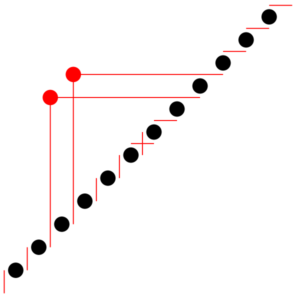
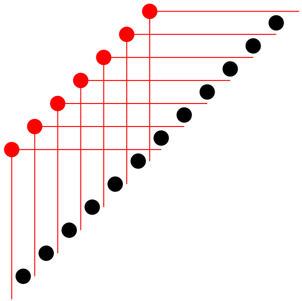


- There are $n + 1$ gaps to fill (including ends).

Two Intervals at a Time



- Need $\left\lceil \frac{n+1}{2} \right\rceil$ additional points.

The Substitution Decomposition

- Every permutation has a block decomposition.
- Gives a unique simple permutation.

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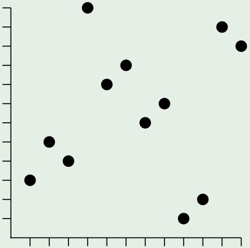

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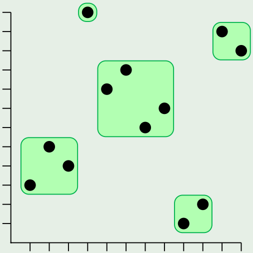

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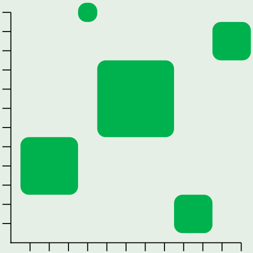

The Substitution Decomposition

- If simple has >2 points then the blocks are unique.
- This is called the substitution decomposition.

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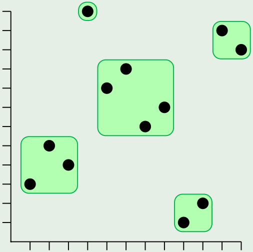

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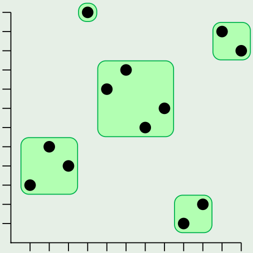
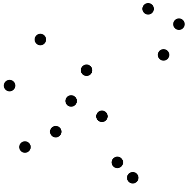


Approach with Induction

Claim: Form two extensions of a permutation of length n.

- At most $\lceil (n+1)/2 \rceil$ additional points each.
- First: new leftmost point and new maximum.
- Second: new leftmost point and new minimum.
- At least one is simple. The other nearly so.



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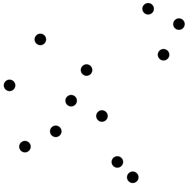

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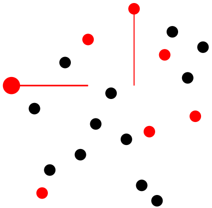

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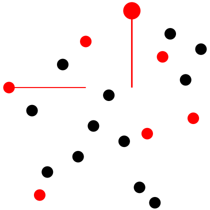

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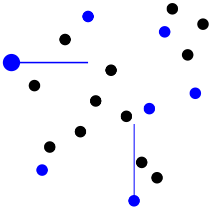

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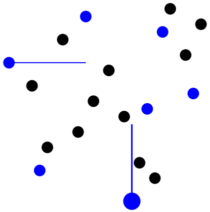

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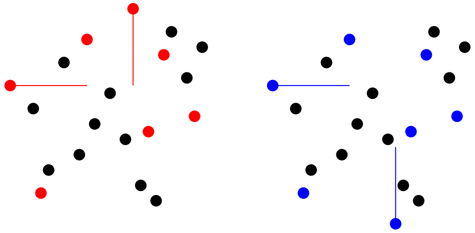
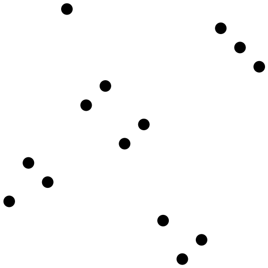


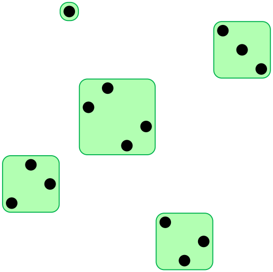
The Inductive Step

- Given a permutation on *n* points.



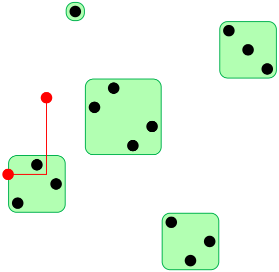
The Inductive Step

- Decompose permutation into smaller blocks.



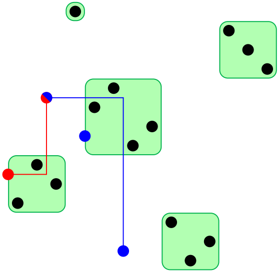
The Inductive Step

- Working left to right, extend each block.



The Inductive Step

- Max / min becomes leftmost point for next block.



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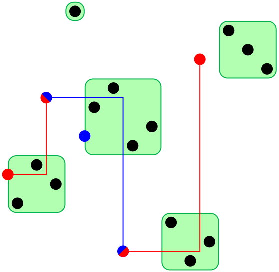
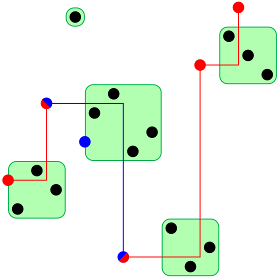


The Inductive Step

- Final block: use max or min.



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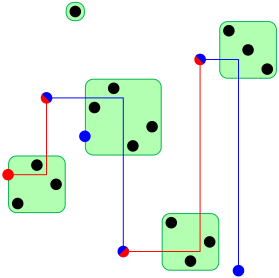

And so...

Theorem (RB, NR, VV)

A permutation on n points has a simple extension requiring at most $\left\lceil \frac{n+1}{2} \right\rceil$ *additional points.*

Outline

The Graph Bound

- Worst cases: complete and independent graphs.

Theorem (Sumner, 1971)

K_n has a simple extension with $\lceil \log_2(n+1) \rceil$ additional vertices.

- Bound is tight.
- General case: use the substitution decomposition.

Theorem (RB, NR, VV)

A graph on n vertices has a simple extension requiring at most $\lceil \log_2(n+1) \rceil$ additional vertices.

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Posets: A Graph-Permutation Mix

- Two (different) bad cases: antichains and linear orders.
- Antichains behave like graphs. Get $\lceil \log_2(n+1) \rceil$.
- Linear orders behave like permutations. Get $\lceil (n+1)/2 \rceil$.
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- Permutations: when do we need all $\lceil (n+1)/2 \rceil$ points?
- Same question for graphs, posets?
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