Rigorous game theory with few prerequisites

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Mind the Gap

The Open University had a 30-credit module 3rd year module on discrete mathematics, running largely unchanged for > 20 years. Much of the existing material could be reworked, but 25% (7.5 credits) needed replacement.

Besides maths students, module taken by engineers, economists, and (increasingly) data scientists. Some have only 30–60 credits of maths at level 1.

Could we meaningfully introduce game theory in a way that is both

(a) accessible to those with minimal mathematical background, and

(b) taught rigorously, as part of a mathematics module?

Structure of new module

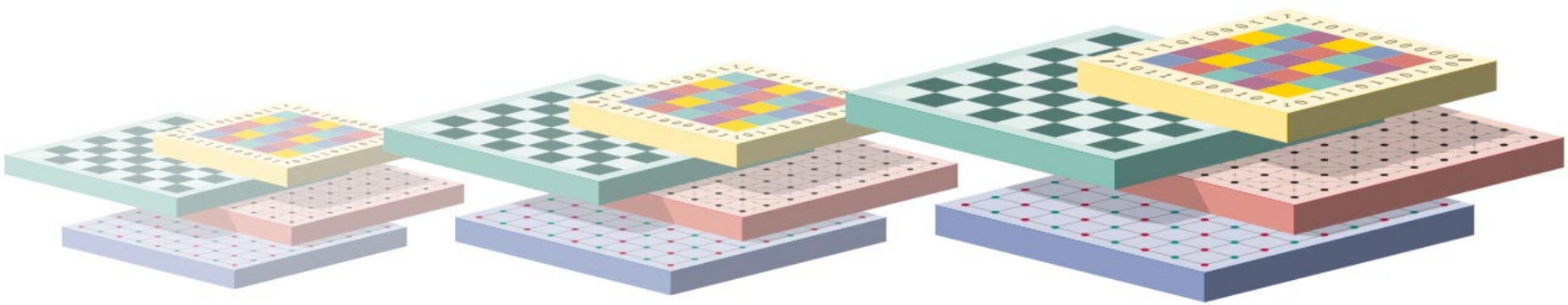
A ‘playful’ blend of applications of algorithmic methods, with rigorous mathematical proofs.

Book A Graph theory

Book B ‘Networks’ (algorithms on graphs and digraphs)

Book C Game theory

Book D Designs (including Latin squares, coding theory, design theory)



Banner image for Graphs, games and designs

Avoid

Calculus

Linear algebra, beyond matrix multiplication

Analysis

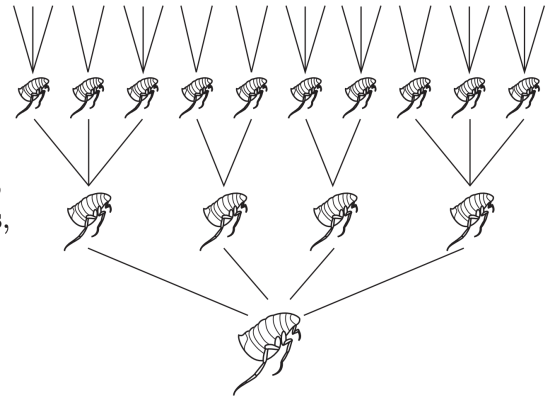
Probability (except what we can teach in the module)

Available

Trees (and graphs)

Basic matrix manipulation

Great fleas have little fleas
upon their backs to bite 'em,
And little fleas have lesser fleas,
and so *ad infinitum*.



Augustus De Morgan, after Jonathan Swift,
as appearing in Graphs, games and designs

Things we *wanted to* include

Combinatorial games, especially impartial ones like Nim

Applications (e.g. economics, elections, evolutionary biology)

Things we *should* include

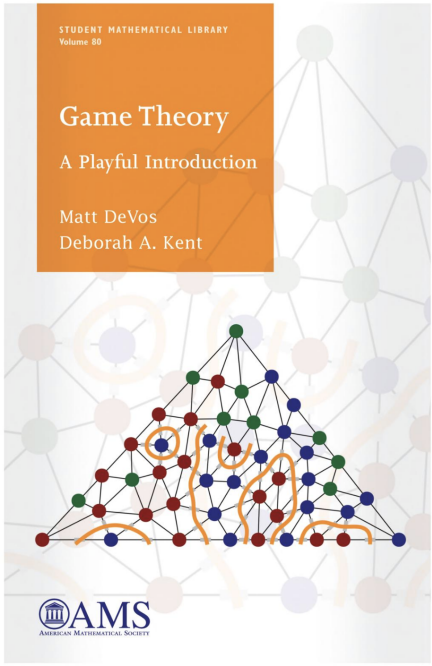
Some basic probability

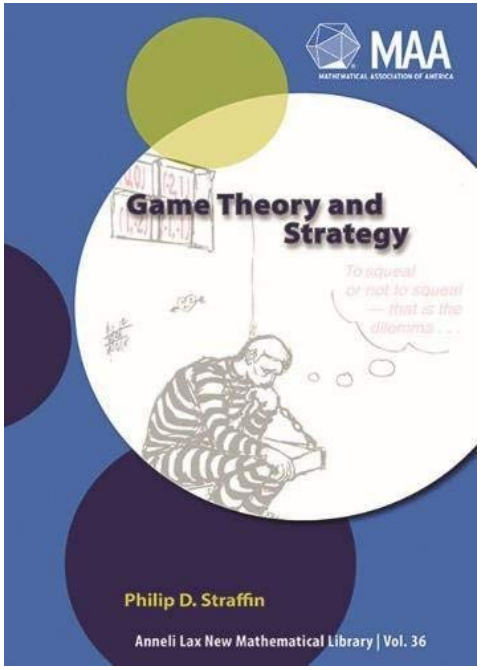
Something about *rational play* and *strategy*

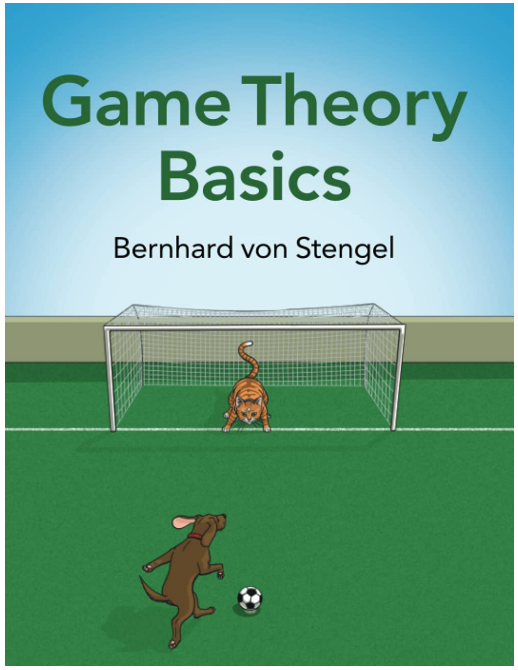
Some ability to solve zero-sum games

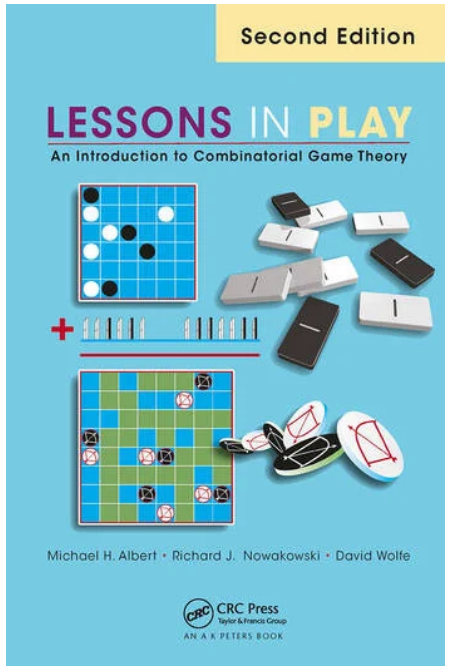
Understanding Nash equilibria, and the Prisoner's dilemma

Textbooks

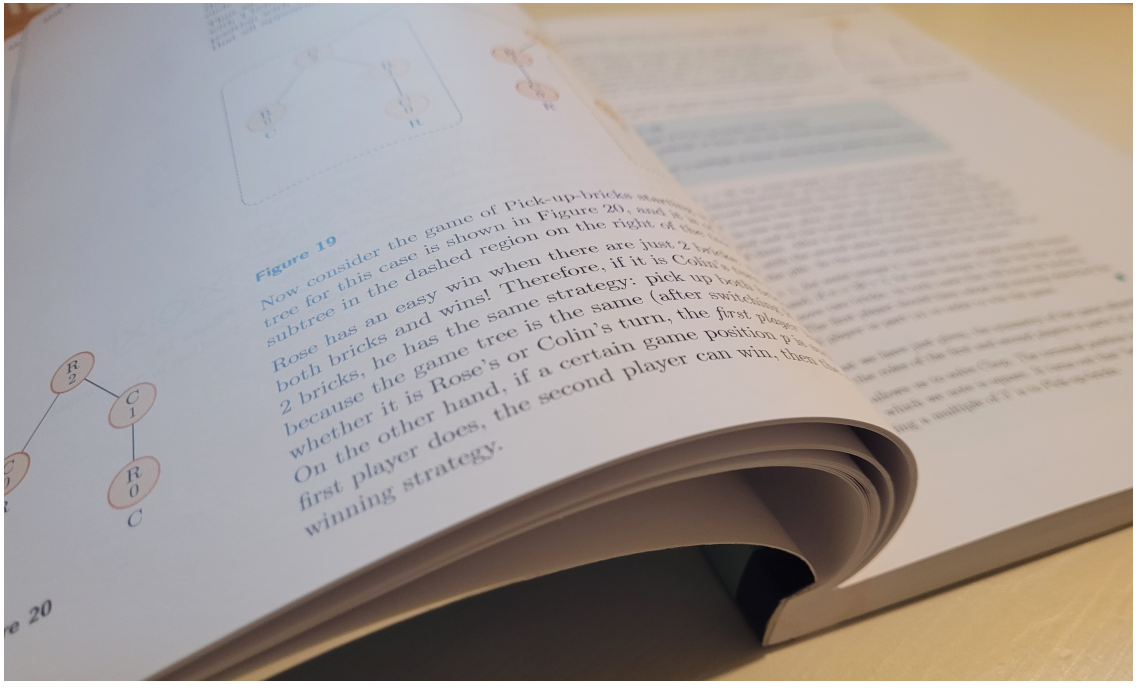

STUDENT MATHEMATICAL LIBRARY
Volume 80
Game Theory
A Playful Introduction
Matt DeVos
Deborah A. Kent
AMS
American Mathematical Society


MAA
Mathematical Association of America
Game Theory and Strategy
Philip D. Straffin
Anneli Lax New Mathematical Library | Vol. 36


Game Theory Basics
Bernhard von Stengel


Second Edition
LESSONS IN PLAY
An Introduction to Combinatorial Game Theory
Michael H. Albert · Richard J. Nowakowski · David Wolfe
CRC Press
Taylor & Francis Group
AN A K PETERS BOOK

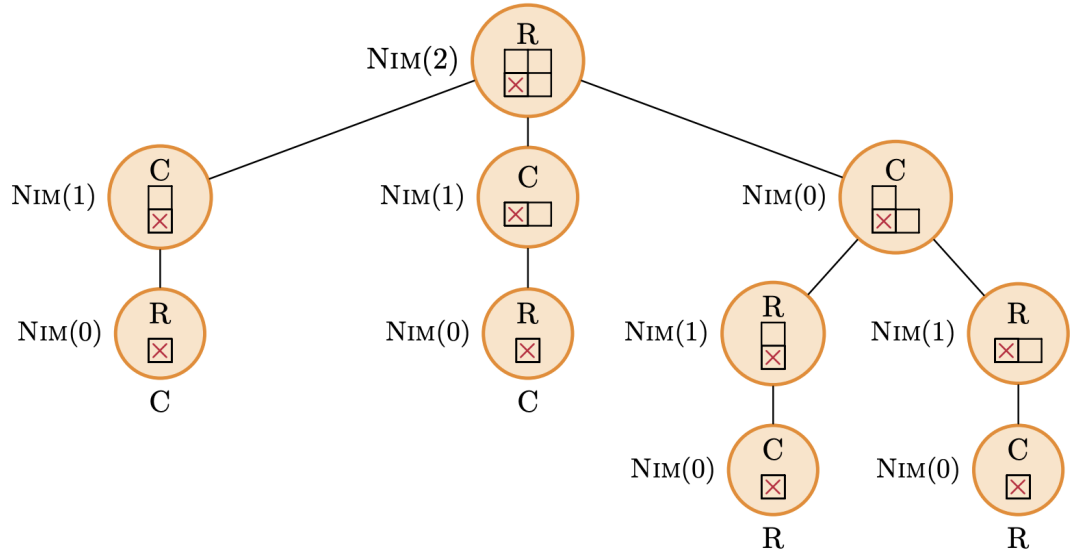
Book C *Game theory* overview



Book C of Graphs, games and designs

Chapter 1: Introduction to games

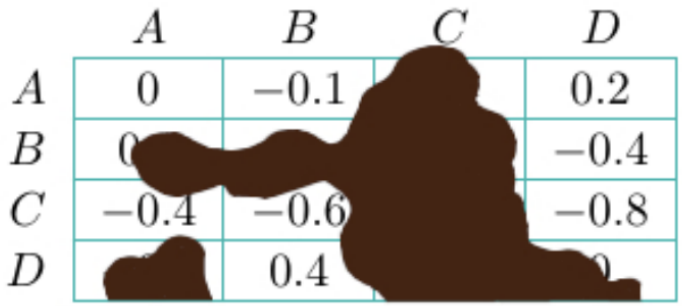
1. Introductory material (‘What is a game?’)
2. Rationality and utility (von Neumann’s lotteries for finding utility functions)
3. Combinatorial games (strategy, drawing game trees)
4. Impartial games (Sprague-Grundy Theorem, mex and nimbers)



A game tree for Chomp with nimbers marked

Chapter 2: Zero-sum games

1. Pure strategies and dominance (easy ways to solve games, saddle points)
2. Mixed strategies (basic probability, von Neumann solutions)
3. Finding von Neumann solutions (including some $2 \times n$ and $m \times 2$ games)
4. Applications (some problem solving, voting systems)

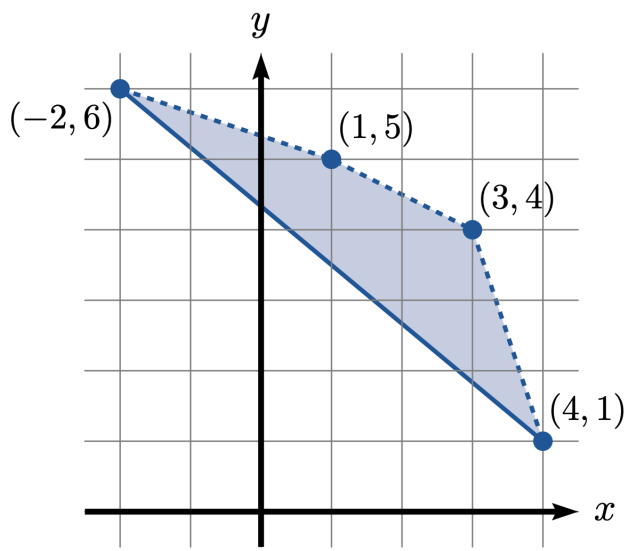


	A	B	C	D
A	0	−0.1		0.2
B	0			−0.4
C	−0.4	−0.6		−0.8
D		0.4		0

A margin table from an election, with coffee spilt over it

Chapter 3: General games and Nash equilibria

1. Introductory material (with connections to zero-sum games)
2. Social vs individual preferences (Prisoner's dilemma, Pareto optimality)
3. Nash equilbria (pure and mixed, best responses, statement of Nash's Theorem)
4. Solving 2×2 games (including proof of Nash's theorem in this case)
5. Applications (Cournot duopoly, evolutionary biology – cleaner fish)



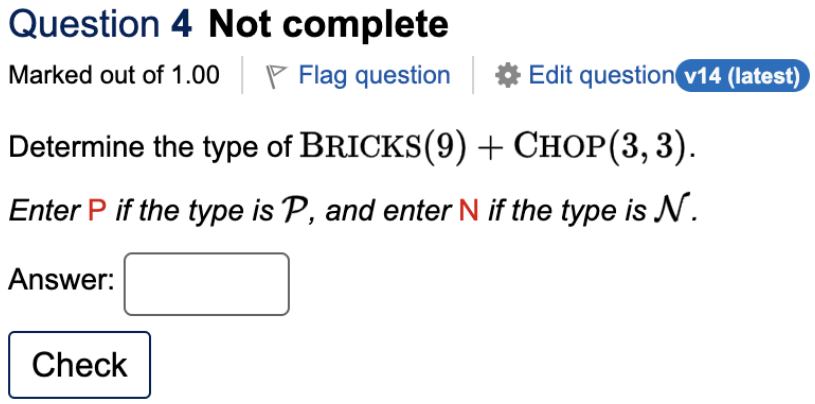
A payoff polygon and the Pareto front

Assessment

3 Tutor-marked assignments Questions are a mixture of ‘use the method to . . . ’ and ‘prove that . . . ’.

Computer-marked assignment 6 computational questions in a ‘revision’ assignment. Written using *Stack*.

Remote exam 3 short questions and one long question. Similar composition to tutor marked assignments.



A Stack question about impartial games

Student feedback (after first presentation, 2023–4)

I really enjoyed Book C [*Game theory*] and thought the explanations were clear and informative. [. . .] Overall, I enjoyed the module very much and if I were choosing my Level 3 modules again, this module would definitely be at the top of my list.

once the basic ideas sank in, the Games topics become reasonably straightforward to grasp and there is a flow from C1 to C2 to C3.

I genuinely think this has been the best module I've studied since leaving school. The balance of maths with its applications, how to think along certain lines, with comments on current practice was extremely well put together.

The game theory section was mainly clear; I think the coverage of "Pareto optimality" could be a bit better explained.

Thanks for your attention!